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ORIGINAL PAPER

Estimates in direct inequalities for the Szász–Mirakyan operator

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Abstract
This paper deals with the approximation of continuous functions by the classical Szász–Mirakyan operator. We give new bounds for the constant in front of the second order Ditzian–Totik modulus of smoothness in direct inequalities. Asymptotic and non asymptotic results are stated. We use both analytical and probabilistic methods, the latter involving the representation of the operators in terms of the standard Poisson process. A smoothing technique based on a modification of the Steklov means is also applied.

Keywords Szász–Mirakyan operators · Ditzian–Totik modulus of smoothness · Direct inequalities · Steklov means

Mathematics Subject Classification 41A17 · 41A35

1 Introduction

The classical Szász–Mirakyan operator S_t, introduced independently in the 1940s by G. M. Mirakyan, J. Favard and O. Szász, associates with each number $t \geq 1$, and each function $f \in \mathbb{R}^{[0,\infty)}$, provided the series below is absolutely convergent, the function $S_t f := S_t(f)$ defined by

$$S_t f(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{t}\right) \frac{e^{-tx}(tx)^k}{k!}, \quad x \geq 0.$$

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This paper deals with the approximation process of $S_t f(x)$ towards $f(x)$, as $t \to \infty$. In this regard, let us introduce some notations. The second order central difference of $f \in \mathbb{R}^{[0,\infty)}$ is defined as

$$\Delta_h^2 f(x) = f(x-h) - 2f(x) + f(x+h), \quad x \geq h \geq 0.$$

The Ditzian–Totik modulus of smoothness of f with step-weight function $\varphi(x) = \sqrt{x}$ is given by

$$\omega_2^\varphi(f;\delta) = \sup\left\{\left|\Delta_{h\sqrt{x}}^2 f(x)\right| : x \geq h^2, \quad 0 \leq h \leq \delta\right\}, \quad \delta \geq 0.$$

Note that the operator S_t interpolates at 0, and holds fixed every affine function. We denote by $\mathfrak{F}$ the set of continuous non affine functions $f \in \mathbb{R}^{[0,\infty)}$ such that $\omega_2^\varphi(f;1) < \infty$. We assume that $t \geq 1$ and denote by $\|f\|_A = \sup\{|f(x)| : x \in A\}$, $A \subseteq [0,\infty)$.

In 1994, Totik [10] obtained the following characterization concerning the rates of uniform convergence for the Szász–Mirakyan operator S_n:

$$\widetilde{K}\omega_2^\varphi\left(f;\frac{1}{\sqrt{n}}\right) \leq \|S_n f - f\|_{[0,\infty)} \leq K\omega_2^\varphi\left(f;\frac{1}{\sqrt{n}}\right), \quad n = 1,2,\ldots, \tag{1}$$

for some absolute positive constants $\widetilde{K}$ and K. The lower and the upper inequalities in (1) are called the converse and the direct inequalities, respectively. Once (1) is established, the subsequent natural question is to estimate the constants $\widetilde{K}$ and K. To the best of our knowledge, the only result in this direction was proved in [2], where it is shown that the direct inequality holds for $K = 4$. It seems that no specific value for $\widetilde{K}$ has been provided yet.

In this paper, we give the following estimates.

Theorem 1 *We have*

$$1 \leq \sup_{t\geq 1}\sup_{f\in\mathfrak{F}} \frac{1}{\omega_2^\varphi\left(f;\frac{1}{\sqrt{t}}\right)}\|S_t f - f\|_{[0,\infty)} \leq 2.43.$$

Theorem 2 *Let* $\tau : [1,\infty) \longrightarrow (0,\infty)$ *be a function such that*

$$\tau(t) \longrightarrow \infty \quad \text{and} \quad \frac{\tau(t)}{t} \longrightarrow 0, \quad \text{as } t \to \infty. \tag{2}$$

Then,

$$\limsup_{t\to\infty} \frac{1}{\omega_2^\varphi\left(f;\frac{1}{\sqrt{t}}\right)}\|S_t f - f\|_{[\tau(t)/t,\infty)} \leq \frac{3}{2}.$$

Theorems 1 and 2 give us, respectively, non-asymptotic and asymptotic estimates of the constant K. As we will see in Sect. 5, the maximum value of $|S_t f(x) - f(x)|$ occurs when the product tx takes intermediate values, specifically, when $tx \in [1,30]$. This is the reason why the asymptotic estimate in Theorem 2 is better than that in Theorem 1.

It may be of interest to compare Theorems 1 and 2 with other known results in the literature referring to the classical Bernstein polynomials. Direct and converse inequalities for such polynomials, analogous to those in (1), were obtained by Ditzian and Ivanov [4], and Totik [9]. Different authors have obtained specific values for the corresponding constant K in the direct inequality. In this regard, Adell and Sangüesa [2] gave $K = 4$, Gavrea et al. [5] and Bustamante [3] provided $K = 3$, and finally, Păltănea [7] obtained $K = 2.5$, this being the best result up to the date and up to our knowledge. Note that Theorem 1 provides $K = 2.43$ for the Szász–Mirakyan operator. Finally, a similar result to Theorem 2 for the Bernstein polynomials was obtained in [1].

Two main different tools are used to prove Theorems 1 and 2. In the first place, the following probabilistic representation of S_t. Let $(N_\lambda)_{\lambda \geq 0}$ be the standard Poisson process. Since

$$P(N_\lambda = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots, \quad \lambda \geq 0, \tag{3}$$

then $S_t f$ can be written as

$$S_t f(x) = \mathbb{E} f\left(\frac{N_{tx}}{t}\right), \tag{4}$$

where $\mathbb{E}$ stands for mathematical expectation. Note that

$$\mathbb{E}\left(\frac{N_{tx}}{t} - x\right) = 0, \qquad \mathbb{E}\left(\frac{N_{tx}}{t} - x\right)^2 = \frac{x}{t}. \tag{5}$$

In the second place, we use a certain smooth approximant $Q_h^a f$ of $f \in \mathfrak{F}$, built up by antisymmetrizing near the origin the classical Steklov means of f (see Sect. 4 for more details).

This paper is organized as follows. In the following section, we include some auxiliary results involving estimates of tail probabilities for the standard Poisson process, as well as other estimates in terms of the Ditzian–Totik modulus. Sections 3 and 4 contain estimates of $|S_t f(x) - f(x)|$ for small and large values of x, respectively. The last section is devoted to prove Theorems 1 and 2.

2 Auxiliary results

In this section, we gather some technical results. Specifically, Lemma 3 was already used in [2] and comes from Petrov [8, p. 52], so that we state it without proof. Lemma 4 is similar to Lemma 2.1 in [2], a proof of which is included here for the sake of completeness. Finally, Lemma 5 is a reformulation of Lemma 2.5.7 in [7], adapted to our setting.

Lemma 3 *Assume that* $0 < a < 1$. *For any* $\lambda > 0$, *we have*

$$P(N_\lambda \leq a\lambda) \leq e^{-\lambda(1-a+a\log a)}.$$

From now on, $\lceil\cdot\rceil$ stands for the ceiling function.

Lemma 4 *Let* $z \geq \epsilon > 0$ *and* $\delta > 0$. *Then,*

$$\left|\Delta_\epsilon^2 f(z)\right| \leq \left\lceil \frac{2\epsilon/\delta^2}{1 + \sqrt{1 + 4(z-\epsilon)/\delta^2}} \right\rceil^2 \omega_2^\varphi(f; \delta).$$

Proof For any $k \in \mathbb{N}$, it can be checked that

$$\Delta_\epsilon^2 f(z) = \sum_{r=1}^{2k-1} a_r \Delta_{\epsilon/k}^2 f(u_r), \quad a_r = \begin{cases} r, & r = 1, \ldots, k-1; \\ 2k - r, & r = k, \ldots, 2k-1, \end{cases} \tag{6}$$

where $u_r = z - \epsilon + r\epsilon/k$. Note that

$$\sum_{r=1}^{2k-1} a_r = k^2. \tag{7}$$

Since φ is increasing, we have, for $r = 1, \ldots 2k-1$,

$$\begin{aligned}\left|\Delta^2_{\epsilon/k} f(u_r)\right| &= \left| f\left(u_r - \varphi(u_r)\frac{\epsilon}{k\varphi(u_r)}\right) - 2f(u_r) + f\left(u_r + \varphi(u_r)\frac{\epsilon}{k\varphi(u_r)}\right)\right| \\ &\le \omega_2^{\varphi}\left(f; \frac{\epsilon}{k\varphi(u_1)}\right).\end{aligned}$$

Thus, we have from (6) and (7),

$$\left|\Delta^2_{\epsilon} f(z)\right| \le k^2 \omega_2^{\varphi}\left(f; \frac{\epsilon}{k\varphi(u_1)}\right).$$

Therefore, the result follows after choosing k to be the smallest integer such that

$$\delta \ge \frac{\epsilon}{k\varphi(u_1)} = \frac{\epsilon}{k\sqrt{z - \epsilon + \epsilon/k}},$$

namely,

$$k = \left\lceil \frac{2\epsilon/\delta^2}{1 + \sqrt{1 + 4(z-\epsilon)/\delta^2}} \right\rceil.$$

□

Lemma 5 *Let $0 \le a < b$, and let $f \in C[0, \infty)$ be such that $f(a) = f(b) = 0$. Then,*

$$\|f\|_{[a,b]} \le \omega_2^{\varphi}\left(f; \frac{b-a}{2\varphi(\frac{a+b}{2})}\right).$$

Proof Suppose that $y \in (a, (a+b)/2]$. Since $f(a) = 0$, we see that

$$\begin{aligned}|f(y)| &= \frac{1}{2}|f(y - (y-a)) - 2f(y) + f(y + y - a) - f(y + y - a)| \\ &\le \frac{1}{2}\left| f\left(y - \varphi(y)\frac{y-a}{\varphi(y)}\right) - 2f(y) + f\left(y + \varphi(y)\frac{y-a}{\varphi(y)}\right)\right| + \frac{1}{2}\|f\|_{[a,b]} \\ &\le \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{y-a}{\varphi(y)}\right) + \frac{1}{2}\|f\|_{[a,b]}.\end{aligned}$$

This and the fact that the function $y \mapsto (y-a)/\varphi(y)$ increases in $(a, (a+b)/2]$, allow us to write

$$|f(y)| \le \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{b-a}{2\varphi(\frac{a+b}{2})}\right) + \frac{1}{2}\|f\|_{[a,b]}. \tag{8}$$

Analogously, for $y \in [(a+b)/2, b)$, since $f(b) = 0$, we have

$$\begin{aligned}|f(y)| &= \frac{1}{2}|-f(y - (b-y)) + f(y - (b-y)) - 2f(y) + f(y + b - y)| \\ &\le \frac{1}{2}\|f\|_{[a,b]} + \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{b-y}{\varphi(y)}\right) \le \frac{1}{2}\|f\|_{[a,b]} + \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{b-a}{2\varphi(\frac{a+b}{2})}\right),\end{aligned} \tag{9}$$

because the function $y \mapsto (b-y)/\varphi(y)$ decreases in $[(a+b)/2, b)$.

The result follows from (8) and (9). $\square$

3 Direct estimates near the origin

We denote by $\Delta_*^1 f$ and $\Delta_*^2 f$ the usual first and second forward differences of f with increment $1/t$, i.e.,

$$\Delta_*^1 f(z) = f\left(z+\frac{1}{t}\right) - f(z), \qquad z \geq 0,$$

$$\Delta_*^2 f(z) = f(z) - 2f\left(z+\frac{1}{t}\right) + f\left(z+\frac{2}{t}\right), \qquad z \geq 0.$$

It turns out that the first and second derivatives of $S_t f$ can be written as

$$(S_t f)'(x) = t S_t\left(\Delta_*^1 f\right)(x) = t\mathbb{E}\Delta_*^1 f\left(\frac{N_{tx}}{t}\right), \tag{10}$$

$$(S_t f)''(x) = t^2 S_t\left(\Delta_*^2 f\right)(x) = t^2\mathbb{E}\Delta_*^2 f\left(\frac{N_{tx}}{t}\right), \tag{11}$$

and the easy to check inequality

$$\left|\Delta_*^2 f\left(\frac{x}{t}\right)\right| \leq \omega_2^\varphi\left(f;\frac{1}{\sqrt{t}}\right), \qquad x \geq 0,$$

yields

$$\left|(S_t f)''(x)\right| \leq t^2\omega_2^\varphi\left(f;\frac{1}{\sqrt{t}}\right), \qquad x \geq 0. \tag{12}$$

Theorem 6 *For $0 \leq x \leq 1/t$, we have, with $\lambda = tx$,*

$$|S_t f(x) - f(x)| \leq \left(1+\frac{\lambda^2}{2}\right)\omega_2^\varphi\left(f;\frac{1}{\sqrt{t}}\right).$$

Proof Let ℓ be the affine function that interpolates f at the points 0 and $1/t$, and let $g := f-\ell$. Since $S_t\ell = \ell$, it goes without saying that

$$S_t g - g = S_t f - f, \quad S_t g(0) = g(0) = g(1/t) = 0, \quad (S_t g)'' = (S_t f)'', \tag{13}$$

and, from (10),

$$(S_t g)'(0) = t(g(1/t) - g(0)) = 0. \tag{14}$$

Let β be a random variable having the beta density $\rho(\theta) = 2(1-\theta)$, $0 \leq \theta \leq 1$. By (11)–(14), and the Taylor's formula, we have

$$\begin{aligned} |S_t g(x)| &= \left|S_t g(0) + x(S_t g)'(0) + \frac{x^2}{2}\mathbb{E}(S_t g)''(x\beta)\right| \\ &= \frac{x^2}{2}\left|\mathbb{E}(S_t f)''(x\beta)\right| \leq \frac{\lambda^2}{2}\omega_2^\varphi\left(f;\frac{1}{\sqrt{t}}\right). \end{aligned} \tag{15}$$

Apply Lemma 5 to g with $a = 0$ and $b = 1/t$ to obtain

$$|g(x)| \leq \omega_2^\varphi\left(g;\frac{1}{\sqrt{2t}}\right) \leq \omega_2^\varphi\left(g;\frac{1}{\sqrt{t}}\right), \qquad 0 \leq x \leq \frac{1}{t}. \tag{16}$$

Finally, from (13)–(16), using the trivial equality $\omega_2^{\varphi}(g;\cdot)=\omega_2^{\varphi}(f;\cdot)$, we have

$$|S_t f(x)-f(x)|=|S_t g(x)-g(x)|\le|S_t g(x)|+|g(x)|\le\left(1+\frac{\lambda^2}{2}\right)\omega_2^{\varphi}\left(f;\frac{1}{\sqrt{t}}\right),$$

and the proof is over. □

4 Direct estimates far from the origin

We will apply a smoothing technique by using the classical second order Steklov mean associated with the function f, which, in probabilistic terms, can be written as

$$P_h f(y)=\mathbb{E}f\left(y+\frac{h}{2}(V_1+V_2)\right),\quad y\ge h>0,$$

where V_1 and V_2 are independent identically distributed random variables having the uniform distribution on $[-1,1]$ (see, for instance, [1] and [6]). Following the lines of the proof of Lemma 2 in [1], it can be seen that

$$|P_h f(y)-f(y)|\le\frac{1}{2}\omega_2^{\varphi}\left(f;\frac{h}{\varphi(y)}\right),\quad y\ge h\ge 0,\tag{17}$$

and

$$\left|(P_h f)''(y)\right|\le\frac{1}{h^2}\omega_2^{\varphi}\left(f;\frac{h}{\varphi(y)}\right),\quad y\ge h\ge 0.\tag{18}$$

Starting from $a,t,x\in\mathbb{R}$, and assuming that

$$0<a\le 1,\qquad \frac{1}{at}\le x,\tag{19}$$

we take

$$h=\frac{\varphi(ax)}{\sqrt{t}}=\frac{\sqrt{ax}}{\sqrt{t}}=\sqrt{a}\sqrt{\mathbb{E}\left(\frac{N_{tx}}{t}-x\right)^2},\tag{20}$$

and define the approximant

$$Q_h^a f(y)=\begin{cases}2P_h f(ax)-P_h f(2ax-y), & 0\le y<ax;\\ P_h f(y), & y\ge ax.\end{cases}\tag{21}$$

Note that (19) and (20) imply that

$$h\le ax,\tag{22}$$

and that $Q_h^a f$ is well defined. Moreover, $Q_h^a f$ is twice differentiable except at the point ax, where it only has sided second derivatives. This implies that its first derivative $\left(Q_h^a f\right)'$ is absolutely continuous, thus allowing us to write, by Taylor's formula,

$$Q_h^a f(y)=Q_h^a f(x)+\left(Q_h^a f\right)'(x)(y-x)+\frac{(y-x)^2}{2}\mathbb{E}\left((Q_h^a f)''(x+(y-x)\beta)\right),\tag{23}$$

with the obvious understanding that $(Q_h^a f)''$ is not properly the second derivative of $Q_h^a f$.

Lemma 7 *Let* $R_a=[ax,\infty)$.

(a) *If* $y \in R_a$*, then*

$$\left|Q_h^a f(y) - f(y)\right| \le \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right),$$

whereas if $y \notin R_a$*, then*

$$\left|Q_h^a f(y) - f(y)\right| \le \left(\frac{3}{2} + \left\lceil \frac{2t(ax-y)}{1+\sqrt{1+4ty}} \right\rceil^2\right)\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right).$$

(b) *For any* $y \ge 0$*, we have*

$$\left|\left(Q_h^a f\right)''(y)\right| \le \frac{1}{h^2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right).$$

Proof (a) If $y \in R_a$, we have from (20)

$$\frac{h}{\varphi(y)} = \frac{\varphi(ax)}{\varphi(y)}\frac{1}{\sqrt{t}} \le \frac{1}{\sqrt{t}}. \tag{24}$$

Thus, the first inequality follows from (17), (21) and (22).

If $y \notin R_a$, we write by virtue of (21)

$$\begin{aligned} Q_h^a f(y) - f(y) &= 2\left(P_h f(ax) - f(ax)\right) - \left(P_h f(2ax-y) - f(2ax-y)\right) \\ &\quad - \left(f(2ax-y) + f(y) - 2f(ax)\right). \end{aligned} \tag{25}$$

Applying Lemma 4 with $z = ax$, $\epsilon = ax - y$ and $\delta = 1/\sqrt{t}$, we have after some simple computations

$$|f(2ax-y) + f(y) - 2f(ax)| = \left|\Delta_{ax-y}^2 f(ax)\right| \le \left\lceil \frac{2t(ax-y)}{1+\sqrt{1+4ty}} \right\rceil^2 \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right).$$

This, together with (17), (24) and (25), shows the second inequality in part (a).

(b) If $y \in R_a$, the result follows from (18) and (21). If $y \notin R_a$, we have from (18), (21) and (24),

$$\left|\left(Q_h^a f\right)''(y)\right| = \left|(P_h f)''(2ax-y)\right| \le \frac{1}{h^2}\omega_2^{\varphi}\left(f; \frac{h}{\varphi(2ax-y)}\right) \le \frac{1}{h^2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right),$$

thus completing the proof. ◻

Denote by 1_A the indicator function of the set A.

Theorem 8 *For* $x \ge 1/(at)$*,* $0 < a \le 1$*, we have, with* $\lambda = tx$*,*

$$|S_t f(x) - f(x)| \le \left(1 + \frac{1}{2a} + \mathbb{E}\left(1 + \left\lceil \frac{2(a\lambda - N_\lambda)}{1+\sqrt{1+4N_\lambda}} \right\rceil^2\right) 1_{\{N_\lambda < a\lambda\}}\right)\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right).$$

Proof Let h be as in (20). We split $S_t f(x) - f(x)$ into three terms:

$$\begin{aligned} S_t f(x) - f(x) &= \left(Q_h^a f(x) - f(x)\right) + \left(S_t f(x) - S_t Q_h^a f(x)\right) + \left(S_t Q_h^a f(x) - Q_h^a f(x)\right) \\ &= I + II + III. \end{aligned} \tag{26}$$

Since $x \ge ax$, Lemma 7*(a)* shows that

$$|I| \le \frac{1}{2}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right). \tag{27}$$

Table 1 Values of λ_i, $i = 1, 2, \ldots, 12$

i	1	2	3	4	5	6	7	8	9	10	11	12
λ_i	1	2.54	5	8	11	13	16	19	22	25	27	30

From (4), and again by Lemma 7*(a)*, we have

$$\begin{aligned} |II| &= \left| \mathbb{E}\left(Q_h^a f\left(\frac{N_{tx}}{t}\right) - f\left(\frac{N_{tx}}{t}\right)\right)\right| \\ &\leq \left(\frac{P\left(N_{tx} \geq atx\right)}{2} + \mathbb{E}\left(\frac{3}{2} + \left\lceil \frac{2(atx - N_{tx})}{1+\sqrt{1+4N_{tx}}}\right\rceil^2\right) 1_{\{N_{tx} < atx\}}\right) \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right) \\ &= \left(\frac{1}{2} + \mathbb{E}\left(1 + \left\lceil \frac{2(a\lambda - N_\lambda)}{1+\sqrt{1+4N_\lambda}}\right\rceil^2\right) 1_{\{N_\lambda < a\lambda\}}\right) \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right). \end{aligned} \tag{28}$$

Finally, from (4), (23), (5), and by Lemma 7(b), we get

$$\begin{aligned} |III| &= \frac{1}{2}\left| \mathbb{E}\left(Q_h^a f\right)''\left(x + \left(N_{tx}/t - x\right)\beta\right)\left(\frac{N_{tx}}{t} - x\right)^2 \right| \\ &\leq \frac{x}{2h^2 t}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right) = \frac{1}{2a}\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right), \end{aligned} \tag{29}$$

where the last equality follows from (20). The result follows from (26)–(29). □

5 Proofs of the main results

Proof of Theorem 1 In the first place, we prove the upper inequality in Theorem 1. Let us consider the following partition of the interval $[0, \infty)$:

$$[0, \infty) = [0, 1/t) \cup \left(\bigcup_{i=1}^{11} \left[\lambda_i/t, \lambda_{i+1}/t\right)\right) \cup [30/t, \infty),$$

where the λ_i's are given in Table 1.

We will prove the result for $x \in [0, \infty)$, separately for each one of the aforesaid subintervals.

For $x \in [0, 1/t)$, Theorem 6 yields

$$|S_t f(x) - f(x)| \leq 1.5\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right). \tag{30}$$

As regards the middle subintervals, we let $i = 1, 2, \ldots, 11$, and apply Theorem 8 with $a = i/\lambda$ and $\lambda = tx$, to obtain

$$\begin{aligned} |S_t f(x) - f(x)| &\leq \left(1 + \frac{\lambda}{2i} + \mathbb{E}\left(1 + \left\lceil \frac{2(i - N_\lambda)}{1+\sqrt{1+4N_\lambda}}\right\rceil^2\right) 1_{\{N_\lambda < i\}}\right) \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right). \\ &\leq \left(1 + \frac{\lambda}{2i} + \sum_{r=0}^{i-1}\left(1 + \left\lceil \frac{2(i-r)}{1+\sqrt{1+4r}}\right\rceil^2\right) e^{-\lambda}\frac{\lambda^r}{r!}\right)\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right) \end{aligned}$$

Table 2 Explicit expressions for $e^{\lambda}(k_i(\lambda) - 1 - \lambda/(2i))$, $i = 1, 2, \ldots, 11$, calculated with the aid of the computer software Mathematica

i	$e^{\lambda}(k_i(\lambda) - 1 - \lambda/(2i))$
1	2
2	$5 + 2\lambda$
3	$10 + 5\lambda + \lambda^2$
4	$17 + 5\lambda + \lambda^2 + \frac{\lambda^3}{3}$
5	$26 + 10\lambda + \frac{5\lambda^2}{2} + \frac{\lambda^3}{3} + \frac{\lambda^4}{12}$
6	$37 + 17\lambda + \frac{5\lambda^2}{2} + \frac{5\lambda^3}{6} + \frac{\lambda^4}{12} + \frac{\lambda^5}{60}$
7	$50 + 17\lambda + 5\lambda^2 + \frac{5\lambda^3}{6} + \frac{5\lambda^4}{24} + \frac{\lambda^5}{60} + \frac{\lambda^6}{360}$
8	$65 + 26\lambda + 5\lambda^2 + \frac{5\lambda^3}{3} + 5\frac{\lambda^4}{24} + \frac{\lambda^5}{24} + \frac{\lambda^6}{360} + \frac{\lambda^7}{2520}$
9	$82 + 26\lambda + \frac{17\lambda^2}{2} + \frac{5\lambda^3}{3} + \frac{5\lambda^4}{24} + \frac{\lambda^5}{24} + \frac{\lambda^6}{360} + \frac{\lambda^7}{2520} + \frac{\lambda^8}{20160}$
10	$101 + 37\lambda + \frac{17\lambda^2}{2} + \frac{17\lambda^3}{6} + \frac{5\lambda^4}{12} + \frac{\lambda^5}{24} + \frac{\lambda^6}{144} + \frac{\lambda^7}{2520} + \frac{\lambda^8}{20160} + \frac{\lambda^9}{181440}$
11	$122 + 50\lambda + 13\lambda^2 + \frac{17\lambda^3}{6} + \frac{5\lambda^4}{12} + \frac{\lambda^5}{12} + \frac{\lambda^6}{144} + \frac{\lambda^7}{1008} + \frac{\lambda^8}{20160} + \frac{\lambda^9}{181440} + \frac{\lambda^{10}}{1814400}$

Table 3 Values of $k_i(\lambda_i)$, $k_i(\lambda_{i+1})$, $i = 1, 2, \ldots, 11$, rounded to six significant figures, calculated with the aid of the computer software Mathematica

i	$k_i(\lambda_i)$	$k_i(\lambda_{i+1})$
1	2.23576	2.42773
2	2.42997	2.35107
3	2.23761	2.37158
4	2.09784	2.38563
5	2.13511	2.30834
6	2.10838	2.33641
7	2.15217	2.35830
8	2.19103	2.37544
9	2.22343	2.38904
10	2.25046	2.35012
11	2.22762	2.36368

$$:= k_i(\lambda)\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right),$$

where explicit expressions for $k_i(\lambda)$, $i = 1, 2, \ldots, 11$, can be easily derived from the information in Table 2.

For $i = 1, 2, \ldots, 11$, the function $k_i(\lambda)$ takes values less than 2.43 at the end points of the interval $[\lambda_i, \lambda_{i+1})$, as Table 3 shows. In addition to that, $k_i(\lambda)$ is convex for $\lambda = tx \in [\lambda_i, \lambda_{i+1})$, since in that interval, one has that $k_i''(\lambda) > 0$. Indeed, it suffices to inspect Table 4 and check that the quantities between curly brackets are positive. For instance, if $i = 6$,

Table 4 Explicit expressions for $e^{\lambda}k_i''(\lambda)$, $i=1,2,\ldots,11$, calculated with the aid of the computer software Mathematica

i	$e^{\lambda}k_i''(\lambda)$
1	2
2	$1+2\lambda$
3	$2+\lambda+\lambda^2$
4	$9+3\lambda+\left\{-\lambda^2+\frac{\lambda^3}{3}\right\}$
5	$11+2\lambda+\frac{3\lambda^2}{2}+\left\{-\frac{\lambda^3}{3}+\frac{\lambda^4}{12}\right\}$
6	$8+12\lambda+\left\{-\frac{3\lambda^2}{2}+\frac{\lambda^3}{2}-\frac{\lambda^4}{12}+\frac{\lambda^5}{60}\right\}$
7	$26+2\lambda+\frac{5\lambda^2}{2}+\left\{-\frac{\lambda^3}{2}+\frac{\lambda^4}{8}-\frac{\lambda^5}{60}+\frac{\lambda^6}{360}\right\}$
8	$23+16\lambda+\left\{-\frac{5\lambda^2}{2}+\frac{5\lambda^3}{6}-\frac{\lambda^4}{8}+\frac{\lambda^5}{40}-\frac{\lambda^6}{360}+\frac{\lambda^7}{2520}\right\}$
9	$47+2\lambda+\lambda^2+\frac{5\lambda^3}{6}+\left\{-\frac{\lambda^4}{8}+\frac{\lambda^5}{40}-\frac{\lambda^7}{2520}+\frac{\lambda^8}{20160}\right\}$
10	$44+20\lambda+\left\{-\frac{7\lambda^2}{2}+\frac{\lambda^3}{3}+\frac{5\lambda^4}{24}-\frac{\lambda^5}{40}+\frac{\lambda^6}{240}-\frac{\lambda^8}{20160}+\frac{\lambda^9}{181440}\right\}$
11	$48+15\lambda+\lambda^2+\frac{7\lambda^3}{6}+\left\{-\frac{5\lambda^4}{24}+\frac{\lambda^5}{24}-\frac{\lambda^6}{240}+\frac{\lambda^7}{1680}-\frac{\lambda^9}{181440}+\frac{\lambda^{10}}{1814400}\right\}$

for $\lambda \in [\lambda_6, \lambda_7) = [13, 16)$, one has

$$k_6''(\lambda) \geq e^{-\lambda}\left(8+12\lambda-\frac{3\cdot 16^2}{2}+\frac{13^3}{2}-\frac{16^4}{12}+\frac{13^5}{60}\right) > 0.$$

Consequently, for $x \in \left[\lambda_i/t, \lambda_{i+1}/t\right)$, $i=1,2,\ldots,11$,

$$|S_t f(x) - f(x)| \leq 2.43\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right). \tag{31}$$

Finally, as for the unbounded subinterval, Theorem 8 with $a = 0.4$ and $\lambda = tx$, together with Lemma 3, yields

$$\begin{aligned}
|S_t f(x) - f(x)| &\leq \left(1+\frac{1}{0.8}+\mathbb{E}\left(1+\left\lceil\frac{2(0.4\lambda - N_\lambda)}{1+\sqrt{1+4N_\lambda}}\right\rceil^2\right)1_{\{N_\lambda<0.4\lambda\}}\right)\omega_2^{\varphi}\left(f;\frac{1}{\sqrt{t}}\right).\\
&\leq \left(2.25+\left(1+\lceil 0.4\lambda\rceil^2\right)P\left(N_\lambda<0.4\lambda\right)\right)\omega_2^{\varphi}\left(f;\frac{1}{\sqrt{t}}\right)\\
&\leq \left(2.25+\left(1+(0.4\lambda+1)^2\right)e^{-\lambda(0.6+0.4\log(0.4))}\right)\omega_2^{\varphi}\left(f;\frac{1}{\sqrt{t}}\right)\\
&:= k_{12}(\lambda)\omega_2^{\varphi}\left(f;\frac{1}{\sqrt{t}}\right).
\end{aligned}$$

By elementary calculus, one sees that $k_{12}(\lambda)$ decreases for $\lambda = tx \in [30, \infty)$, and also that $k_{12}(30) = 2.40432$ (rounded to six significant figures).

Hence, for $x \in [30/t, \infty)$,

$$|S_t f(x) - f(x)| \leq 2.41\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right). \tag{32}$$

Thus, the upper inequality in Theorem 1 follows from (30), (31) and (32).

To prove the lower inequality, we proceed as in the proof of Theorem 2 in [1]. Let $x \in (0, 1/t)$ and consider the function

$$f_x(y) = \left(1 - \frac{y}{x}\right) 1_{[0,x]}(y).$$

Observe that $\omega_2^{\varphi}(f_x; 1/\sqrt{t}) = 1$, as well as

$$S_t f_x(x) - f_x(x) = \mathbb{E} f_x\left(\frac{N_{tx}}{t}\right) = P(N_{tx} = 0) = e^{-tx}.$$

This implies that

$$\sup_{t \geq 1} \sup_{f \in \mathfrak{F}} \frac{1}{\omega_2^{\varphi}(f; 1/\sqrt{t})} \|S_t f - f\|_{[0,\infty)} \geq e^{-x}.$$

Hence, the lower inequality follows by letting $x \to 0$. The proof is complete. □

Proof of Theorem 2. Let $0 < a < 1$, and let $x \geq \tau(t)/t \geq 1/(at)$. From Theorem 8 and Lemma 5, we have, with $\lambda = tx$,

$$\begin{aligned} |S_t f(x) - f(x)| &\leq \left(1 + \frac{1}{2a} + \left(1 + \lceil a\lambda \rceil^2\right) P\left(N_\lambda < a\lambda\right)\right) \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right) \\ &\leq \left(1 + \frac{1}{2a} + \left(1 + \lceil a\lambda \rceil^2\right) e^{-\lambda(1-a+a\log a)}\right) \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right). \end{aligned} \tag{33}$$

Observe that

$$\|S_t f - f\|_{[\tau(t)/t,\infty)} = \sup_{\lambda \geq \tau(t)} |S_t f(x) - f(x)|.$$

We therefore have from (33) and assumption (2)

$$\limsup_{t \to \infty} \frac{1}{\omega_2^{\varphi}\left(f; \frac{1}{\sqrt{t}}\right)} \|S_t f - f\|_{[\tau(t)/t,\infty)} \leq 1 + \frac{1}{2a}.$$

The proof is over, since $0 < a < 1$ is arbitrary. □

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