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Consistency of hesitant fuzzy linguistic preference relations: An interval consistency index

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ABSTRACT

The study of hesitant consistency is very important in decision-making with hesitant fuzzy linguistic preference relations (HFLPRs), and generally the normalization method is used as a tool to measure the consistency degree of a HFLPR. In this paper we propose a new hesitant consistency measure, called interval consistency index, to estimate the consistency range of a HFLPR. The underlying idea of the interval consistency index consists of measuring the worst consistency index and the best consistency index of a HFLPR. Furthermore, by comparative study, a connection is shown between the interval consistency index and the normalization method, demonstrating that the normalization method should be considered as an approximate average consistency index of a HFLPR.

1. Introduction

Solving a decision problem with linguistic information implies the need for computing with words (CW)[11,23,24,35–37]. In particular, Herrera and Martínez [14] proposed the 2-tuple linguistic representation model, which has been successfully used in a wide range of applications (e.g., [20,21,29,31]). In recent years, different models based on linguistic 2-tuples, such as the proportional 2-tuple linguistic representation model [32], the model based on the linguistic hierarchy [7,12,13] and the numerical scale model [4,8,17], have been presented in the literature.

Generally, when using linguistic models in decision making problems, experts provide a single term to express their preferences [10,14,19]. However, in some situations, experts may prefer to think of several terms at the same time to provide their preferences instead of a single linguistic term [23,26].

To overcome this limitation, Rodríguez et al. [27] introduced the concept of Hesitant Fuzzy Linguistic Term Set (HFLTS) to serve as the basis of increasing the flexibility of the elicitation of linguistic information by means of linguistic expressions. Wei et al. [33] and Dong et al. [7] gave possibility degree formulas to compare HFLTSs and also presented two new linguistic aggregation operators for HFLTSs. Dong et al. [3] proposed an optimization-based consensus model to minimize adjusted simple terms in the consensus reaching process with hesitant linguistic assessments in group decision making. To clearly

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demonstrate the use of hesitant fuzzy sets in decision making, Rodríguez et al. [25,28] presented a complete review on hesitant fuzzy sets and recent results on HFLTS.

It is well known that quantifying consistency is a crucial issue in decision-making with preference relations [5,15,16,18]. In this paper, we focus on the study of measuring the consistency of Hesitant Fuzzy Linguistic Preference Relations (HFLPRs). The two following studies detected drawbacks in quantifying the consistency of HFLPRs have led us to the following proposals:

(1) In [39], Zhu and Xu studied the additive consistency measure of HFLPRs, which we call the normalization method. The normalization method introduces a parameter to add new linguistic terms in order to construct the normalized HFLPR. We can derive many linguistic preference relations based on a HFLPR (see Definition 10). However, the normalization method measures several (not all) linguistic preference relations associated with a HFLPR. Thus, the internal mechanism of the normalization method is not clear due to this partial measurement.
(2) It is natural that the best and worst consistency degrees of the linguistic preference relations associated with the HFLPR play an important role in analyzing the consistency of a HFLPR. However, current studies have not yet studied the measurement of these two consistency degrees in HFLPRs.

In order to overcome previous shortcomings, this paper develops an interval consistency index (ICI) of a HFLPR based on the 2-tuple linguistic model. To do so, the following points are considered:

- An optimization-based model to measure the ICI of a HFLPR is proposed. The underlying idea of the ICI consists of measuring the worst consistency index (WCI) and the best consistency index (BCI) of a HFLPR. Besides, we propose an approach based on the mixed 0–1 linear programming to obtain the optimum solution to the optimization-based model.
- A numerical analysis is provided to illustrate the essence of the normalized consistency index (NCI) presented in Zhu and Xu [39]. Furthermore, by analyzing the average consistency index (ACI) of all linguistic preference relations associated with a HFLPR, the NCI can be seen to reflect the ACI approximately.
- Finally, the difference between the ICI and NCI (or ACI) is analyzed, and the reason for their different behaviors when measuring the consistency degree of HFLPRs has been shown. From such analysis, it has been concluded that the combined use of the ICI and NCI (or ACI) reflect better the consistency status of HFLPRs.

The rest of the paper is organized as follows. Section 2 introduces some basic knowledge. Next, Section 3 presents the ICI via an optimization-based approach to estimate the consistency degree in a HFLPR. Then, in Section 4 a detailed comparative study of different consistency measures is provided. Finally, concluding remarks are included in Section 5.

2. Preliminaries

In this section, we introduce the basic knowledge regarding the 2-tuple linguistic model, HFLTS, linguistic preference relation, HFLPR and the normalization method.

2.1. The 2-tuple linguistic model and hesitant fuzzy linguistic term set

The basic notations and operational laws of linguistic variables were introduced in [35]. Let $S = \{s_j | j = 0, \cdots, g\}$ be a linguistic term set with odd granularity $g + 1$, where the term s_j represents a possible value for a linguistic variable. The linguistic term set is usually required to satisfy the following additional characteristics:

(1) The set is ordered: $s_i \le s_j$ if and only if $i \le j$;
(2) There is a negation operator: $Neg(s_j) = s_{g-j}$.

The 2-tuple linguistic representation model, presented by Herrera and Martínez [14] represents the linguistic information by a 2-tuple $(s_i, \alpha) \in \bar{S} = S \times [-0.5, 0.5)$, where $s_i \in S$ and $\alpha \in [-0.5, 0.5)$. Formally, let $S = \{s_i | i = 0, 1, 2, \ldots, g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation operation. The 2-tuple that expresses the equivalent information to β is then obtained as:

$$\Delta : [0, g] \to S \times [-0.5, 0.5), \tag{1}$$

where

$$\Delta(\beta) = (s_i, \alpha), \quad \text{with} \begin{cases} s_i, & i = \text{round}(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{cases} . \tag{2}$$

Function Δ is a one to one mapping function whose inverse function $\Delta^{-1} : \bar{S} \to [0, g]$ is defined as $\Delta^{-1}(s_i, \alpha) = i + \alpha$. When $\alpha = 0$ in (s_i, α) is then called a simple term.

In [14] a computational model was also defined for the 2-tuple linguistic model in which different operations were introduced:

(1) A 2-tuple comparison operator: Let (s_k, α) and (s_l, γ) be two 2-tuples. Then:
  (i) if $k < l$, then (s_k, α) is smaller than (s_l, γ).

(ii) if $k = l$, then
 (a) if $\alpha = \gamma$, then (s_k, α), (s_l, γ) represents the same information.
 (b) if $\alpha < \gamma$, then (s_k, α) is smaller than (s_l, γ).
 (2) A 2-tuple negation operator:
 $Neg((s_i, \alpha)) = \Delta(g - (\Delta^{-1}(s_i, \alpha)))$.
 (3) Several 2-tuple aggregation operators have been developed (see [14,22]).

Torra [30] proposed the hesitant fuzzy sets. Based on the hesitant fuzzy sets and the fuzzy linguistic approach, the concept of a HFLTS is introduced in Rodríguez et al. [27], as Definition 1.

Definition 1. [27] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set. A HFLTS, H_S, is an ordered finite subset of the consecutive linguistic terms of S.

Definition 2. [27] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set. Let H_S^1 and H_S^2 be two HFLTSs on S,

(1) The intersection $H_S^1 \bigcap H_S^2$ of H_S^1 and H_S^2 is defined by $H_S^1 \bigcap H_S^2 = \{s_k | s_k \in H_S^1 \text{ and } s_k \in H_S^2\}$;
(2) The union $H_S^1 \bigcup H_S^2$ of H_S^1 and H_S^2 is defined by $H_S^1 \bigcup H_S^2 = \{s_k | s_k \in H_S^1 \text{ or } s_k \in H_S^2\}$.

Definition 3. [27] Let H_S be a HFLTS of S. Let $H_S^- = \min_{s_i \in H_S}(s_i)$, $H_S^+ = \max_{s_i \in H_S}(s_i)$ and $env(H_S) = [H_S^-, H_S^+]$. Then, H_S^-, H_S^+ and $env(H_S)$ are called the lower bound, the upper bound and the envelope of H_S.

2.2. Linguistic preference relation and its consistency index

Let $A = \{A_1, A_2, \ldots, A_n\} (n \geq 2)$ be a finite set of alternatives. When a decision maker makes pairwise comparisons using the linguistic term set S, they can construct a linguistic preference relation $L = (l_{ij})_{n\times n}$, whose element l_{ij} estimates the preference degree of alternative A_i over A_j. Linguistic preference relations based on linguistic 2-tuple can be formally defined as in Definition 4.

Definition 4. [1,2] The matrix $L = (l_{ij})_{n\times n}$, where $l_{ij} \in S$, is called a linguistic preference relation. The matrix $L = (l_{ij})_{n\times n}$, where $l_{ij} \in \overline{S}$, is called a 2-tuple linguistic preference relation. If $l_{ij} = Neg(l_{ji})$ for $i, j = 1, 2, \ldots, n$, then L is considered reciprocal.

The additive transitivity is used to character the consistency of linguistic preference relations as in Definition 5.

Definition 5. [2,6] Let $L = (l_{ij})_{n\times n}$ be a linguistic preference relation based on S. L is considered consistent if $\Delta^{-1}(l_{ij}) + \Delta^{-1}(l_{jk}) - \Delta^{-1}(l_{ik}) = \frac{g}{2}$ for $i, j, k = 1, 2, \ldots, n$.

Based on Definition 5, the consistency index (CI) of a linguistic preference relation L can be developed using the Manhattan distance and the Euclidean distance [1,6,8,39], respectively.

Let $\varepsilon_{ijk} = \Delta^{-1}(l_{ij}) + \Delta^{-1}(l_{jk}) - \Delta^{-1}(l_{ik}) - \frac{g}{2}$. Then, the CI using the Manhattan distance is defined as follows,

$$CI(L) = 1 - \frac{2}{3gn(n-1)(n-2)} \sum_{i,j,k=1}^{n} |\varepsilon_{ijk}| \tag{3}$$

The CI using the Euclidean distance is defined as follows,

$$CI(L) = 1 - \frac{2}{3g} \sqrt{\frac{1}{n(n-1)(n-2)} \sum_{i,j,k=1}^{n} (\varepsilon_{ijk})^2} \tag{4}$$

The larger the value of $CI(L)$ the more consistent L is. If $CI(L) = 1$, then L is a consistent linguistic preference relation.

2.3. Hesitant fuzzy linguistic preference relation and the normalization method

Based on the use of HFLTSs, Zhu and Xu [39] proposed the HFLPR as in Definition 6.

Definition 6. Let M_S be a set of HFLTSs based on S. A HFLPR based on S is presented by the matrix $H = (H_{ij})_{n\times n}$, where $H_{ij} \in M_S$ and $Neg(H_{ij}) = H_{ji}$ [39].

When operating with HFLTSs, in order to make sure all have the same number of linguistic terms, Zhu and Xu [39] proposed a principle for normalization: the α-normalization and the β-normalization.

(1) α-normalization: Removes some elements of the HFLTS, which has a higher number of elements.
(2) β-normalization: Adds some elements to the HFLTS, which has fewer elements.

In this paper, we discuss the β-normalization [39] although the results for the α-normalization are similar.

Based on the β-normalization, Zhu and Xu [39] introduced a method to add linguistic terms to HFLTSs to maintain the same number of all HFLTSs in a HFLPR (see Definitions 7 and 8).

Definition 7. [39] Assume a HFLTS, $H_S = \{H_S^q | q = 1, \dots, \#H_S\}$. Let H_S^+ and H_S^- be the maximum and minimum linguistic terms in H_S, respectively, and $\varsigma(0 \le \varsigma \le 1)$ be an optimized parameter, then the term $H_S' = \varsigma H_S^+ + (1-\varsigma)H_S^-$ is called an added linguistic term.

Definition 8. [39] Let $H = (H_{ij})_{n\times n}$ be a HFLPR and $\varsigma(0 \le \varsigma \le 1)$ be an optimized parameter. Using ς to add linguistic terms in $H_{ij}(i<j)$ and $1-\varsigma$ to add linguistic terms in $H_{ji}(i<j)$, the normalized HFLPR with ς, $H^N = (H_{ij}^N)_{n\times n}$, can be obtained, in which

$\#H_{12}^N = \#H_{13}^N = \ldots = \#H_{1n}^N = \ldots = \#H_{ij}^N = \ldots = \#H_{(n-2)n}^N = \#H_{(n-1)n}^N$, $i \neq j$, where $\#H_{ij}^N$ is the number of linguistic terms in H_{ij}^N.

Example 1. Let S be a linguistic term set which is defined as follows:

$S = \{s_0 = extremely\ poor, s_1 = very\ poor, s_2 = poor, s_3 = slightly\ poor, s_4 = fair,$
$s_5 = slightly\ good, s_6 = good, s_7 = very\ good, s_8 = extremely\ good\}.$

Consider the following HFLPR:

$$H = \begin{pmatrix} \{s_4\} & \{s_5, s_6\} & \{s_2, s_3\} & \{s_6\} \\ \{s_3, s_2\} & \{s_4\} & \{s_3, s_4\} & \{s_4, s_5, s_6\} \\ \{s_6, s_5\} & \{s_5, s_4\} & \{s_4\} & \{s_6, s_7, s_8\} \\ \{s_2\} & \{s_4, s_3, s_2\} & \{s_2, s_1, s_0\} & \{s_4\} \end{pmatrix}.$$

Suppose $\varsigma = 1$, then we can transform H into the normalized HFLPR H^N as follows,

$$H^N = \begin{pmatrix} \{s_4\} & \{s_5, s_6, s_6\} & \{s_2, s_3, s_3\} & \{s_6, s_6, s_6\} \\ \{s_3, s_2, s_2\} & \{s_4\} & \{s_3, s_4, s_4\} & \{s_4, s_5, s_6\} \\ \{s_6, s_5, s_5\} & \{s_5, s_4, s_4\} & \{s_4\} & \{s_6, s_7, s_8\} \\ \{s_2, s_2, s_2\} & \{s_4, s_3, s_2\} & \{s_2, s_1, s_0\} & \{s_4\} \end{pmatrix}.$$

Let $H^N = (H_{ij}^N)_{n\times n}$ be a normalized HFLPR. Let $c^N = \#H_{ij}^N (i, j = 1, 2, \ldots, n;\ i \neq j)$ be the number of linguistic terms in H_{ij}^N and let $H_{ij}^N = \{H_{ij}^{N,\rho} | \rho = 1, 2, \ldots, c^N\}$ be the set of all linguistic terms in H_{ij}^N. For example, if $H_{12}^N = \{s_4, s_5\}$, then $H_{12}^{N,1} = s_4$, $H_{12}^{N,2} = s_5$ and $c^N = 2$.

Definition 9 ([39]). Assume a HFLPR H, and its normalized HFLPR H^N with an optimized parameter $\varsigma(0 \le \varsigma \le 1)$. Based on Eq. (4), the NCI of H using the Euclidean distance is defined as follows,

$$NCI(H) = 1 - \frac{2}{3g}\sqrt{\frac{1}{n(n-1)(n-2)} \times \frac{1}{c^N}\sum_{\rho=1}^{c^N}\sum_{i,j,k=1}^{n}(\varepsilon_{ijk})^2}. \tag{5}$$

where $\varepsilon_{ijk} = \Delta^{-1}(H_{ij}^{N,\rho}) + \Delta^{-1}(H_{jk}^{N,\rho}) - \Delta^{-1}(H_{ik}^{N,\rho}) - \frac{g}{2}$.

Similarly, on the basis of Eq. (3), the NCI of H using the Manhattan distance is

$$NCI(H) = 1 - \frac{2}{3gn(n-1)(n-2)} \times \frac{1}{c^N}\sum_{\rho=1}^{c^N}\sum_{i,j,k=1}^{n}|\varepsilon_{ijk}|. \tag{6}$$

The larger the value of $NCI(H)$, the more consistent H *is*. In computing NCI, the optimized parameter ς and the optimized NCI of H are obtained by the following model,

$$\begin{cases} \max\ NCI(H) \\ s.t.\quad 0 \le \varsigma \le 1 \end{cases} \tag{7}$$

3. Interval consistency measure for hesitant fuzzy linguistic preference relations

In this section, we propose an interval consistency measure via a mixed 0–1 linear programming model with the aim of measuring the consistency degree of HFLPRs.

3.1. The approach to obtain the interval consistency level of HFLPRs

Before introducing the interval consistency measure, we define the concept of the linguistic preference relations associated with a HFLPR.

Definition 10. Let $H=(H_{ij})_{n\times n}$ be a HFLPR. $L=(l_{ij})_{n\times n}$ is called a linguistic preference relation associated with H, if $l_{ij}\in H_{ij}$ and $l_{ij}=Neg(l_{ji})$.

We denote N_H as the set of all the linguistic preference relations associated with H.

In the following, we propose an interval consistency measure to estimate the consistency degree of a HFLPR. The underlying idea of the ICI consists in measuring the worst consistency index of the HFLPR H, denoted as $WCI(H)$, and the best consistency index of H, denoted as $BCI(H)$.

Definition 11. Let $H=(H_{ij})_{n\times n}$ be a HFLPR and $L=(l_{ij})_{n\times n}\in N_H$ be the linguistic preference relations associated with H. The ICI of H is,

$$ICI(H)=[WCI(H),BCI(H)] \tag{8}$$

The WCI of H is,

$$WCI(H)=\min_{L\in N_H} CI(L) \tag{9}$$

The BCI of H is,

$$BCI(H)=\max_{L\in N_H} CI(L) \tag{10}$$

The value $WCI(H)$ is determined by its linguistic preference relation with the worst consistency degree, and the value $BCI(H)$ is determined by its linguistic preference relation with the best consistency degree. Thus, the larger the value of $WCI(H)$ and $BCI(H)$, the more consistent H is.

Following, the optimization-based model regarding WCI and BCI is constructed.

Based on Definition 10, $L=(l_{ij})_{n\times n}\in N_H$ equals

$$\begin{cases} l_{ij}\in H_{ij} \\ l_{ij}=Neg(l_{ji}) \end{cases} \tag{11}$$

Thus, if the Manhattan distance is used (i.e., Eq. (3)) to compute the CI of linguistic preference relations, then Eq. (9) can be equivalently transformed into the following model (12)–(14),

$$\min_{L\in N_H} 1-\frac{2}{3gn(n-1)(n-2)}\sum_{i,j,k=1}^{n}|\Delta^{-1}(l_{ij})+\Delta^{-1}(l_{jk})-\Delta^{-1}(l_{ik})-\frac{g}{2}| \tag{12}$$

$$s.t.\quad l_{ij}\in H_{ij} \tag{13}$$

$$l_{ij}=Neg(l_{ji}) \tag{14}$$

Similarly, Eq. (10) can be equivalently transformed into the following model (15)–(17),

$$\max_{L\in N_H} 1-\frac{2}{3gn(n-1)(n-2)}\sum_{i,j,k=1}^{n}|\Delta^{-1}(l_{ij})+\Delta^{-1}(l_{jk})-\Delta^{-1}(l_{ik})-\frac{g}{2}| \tag{15}$$

$$s.t.\quad l_{ij}\in H_{ij} \tag{16}$$

$$l_{ij}=Neg(l_{ji}) \tag{17}$$

To solve the above optimization-based models, a mixed 0–1 linear programming to obtain the optimum consistency solution is proposed. Let $H=(H_{ij})_{n\times n}$ be a HFLPR of S, where $H_{ij}=\{H_{ij}^1,\ldots,H_{ij}^{\#H_{ij}}\}$. We introduce the 0–1 variables as follows,

$$x_{ij}^r=\begin{cases} 0 & if\ l_{ij}\neq H_{ij}^r \\ 1 & if\ l_{ij}=H_{ij}^r \end{cases}\quad i,j=1,2,\ldots,n;\ r=1,\ldots,\#H_{ij}.$$

Clearly, $x_{ij}^r\in\{0,1\}$ and $\sum_{r=1}^{\#H_{ij}}x_{ij}^r=1$.

In this way, $l_{ij}\in H_{ij}$ can be equivalently expressed by x_{ij}^r. For example, suppose $H_{12}=\{H_{12}^1,H_{12}^2,H_{12}^3\}=\{s_1,s_2,s_3\}$. If $\{x_{12}^1,x_{12}^2,x_{12}^3\}=\{0,0,1\}$, then $l_{12}=s_3$; on the contrary, if $l_{12}=s_1$, then $\{x_{12}^1,x_{12}^2,x_{12}^3\}=\{1,0,0\}$.

Lemma 1. *For any $H_{ij}\neq null$ and $l_{ij}\in H_{ij}$, $\Delta^{-1}(l_{ij})=\sum_{r=1}^{\#H_{ij}}(x_{ij}^r\times\Delta^{-1}(H_{ij}^r))$, if $x_{ij}^r\in\{0,1\}$ and $\sum_{r=1}^{\#H_{ij}}x_{ij}^r=1$.*

Proof. Without loss of generality, we assume that $l_{ij}=H_{ij}^k\in H_{ij}$. For $x_{ij}^r\in\{0,1\}$ and $\sum_{r=1}^{\#H_{ij}}x_{ij}^r=1$, we obtain

$\Delta^{-1}(l_{ij})=\Delta^{-1}(H_{ij}^k)=x_{ij}^k\times\Delta^{-1}(H_{ij}^k)+\sum_{r=1,r\neq k}^{\#H_{ij}}(x_{ij}^r\times\Delta^{-1}(H_{ij}^r))=\sum_{r=1}^{\#H_{ij}}(x_{ij}^r\times\Delta^{-1}(H_{ij}^r))$.

As a result, $\Delta^{-1}(l_{ij})=\sum_{r=1}^{\#H_{ij}}(x_{ij}^r\times\Delta^{-1}(H_{ij}^r))$.

This completes the proof of Lemma 1. □

Theorem 1. *By introducing the 0–1 variable* $x_{ij}^r = \begin{cases} 0 & if\ l_{ij} \neq H_{ij}^r \\ 1 & if\ l_{ij} = H_{ij}^r \end{cases}$ $(i, j = 1, 2, \ldots, n; r = 1, \ldots, \#H_{ij})$*. Model (12)–(14) and model (15)–(17) can be equivalently transformed into model (18)–(22) and model (23)–(27):*

$$\min 1 - \frac{2}{3gn(n-1)(n-2)} \sum_{i,j,k=1}^{n} \left| z_{ij} + z_{jk} - z_{ik} - \frac{g}{2} \right| \tag{18}$$

$$s.t. \quad z_{ij} = \sum_{r=1}^{\#H_{ij}} (x_{ij}^r \times \Delta^{-1}(H_{ij}^r)) \quad i, j = 1, 2, \ldots, n \tag{19}$$

$$z_{ji} = g - z_{ij} \qquad i, j = 1, 2, \ldots, n \tag{20}$$

$$x_{ij}^r \in \{0, 1\} \qquad i, j = 1, 2, \ldots, n; r = 1, \ldots, \#H_{ij} \tag{21}$$

$$\sum_{r=1}^{\#H_{ij}} x_{ij}^r = 1 \qquad i, j = 1, 2, \ldots, n \tag{22}$$

and

$$\max 1 - \frac{2}{3gn(n-1)(n-2)} \sum_{i,j,k=1}^{n} \left| z_{ij} + z_{jk} - z_{ik} - \frac{g}{2} \right| \tag{23}$$

$$s.t. \ z_{ij} = \sum_{r=1}^{\#H_{ij}} (x_{ij}^r \times \Delta^{-1}(H_{ij}^r)) \quad i, j = 1, 2, \ldots, n \tag{24}$$

$$z_{ji} = g - z_{ij} \qquad i, j = 1, 2, \ldots, n \tag{25}$$

$$x_{ij}^r \in \{0, 1\} \qquad i, j = 1, 2, \ldots, n; r = 1, \ldots, \#H_{ij} \tag{26}$$

$$\sum_{r=1}^{\#H_{ij}} x_{ij}^r = 1 \qquad i, j = 1, 2, \ldots, n \tag{27}$$

Proof. Based on Lemma 1, we have $\Delta^{-1}(l_{ij}) = z_{ij} = \sum_{r=1}^{\#H_{ij}} (x_{ij}^r \times \Delta^{-1}(H_{ij}^r))$. The constraint $z_{ji} = g - z_{ij}$ guarantees that $l_{ij} = Neg(l_{ji})$. Besides, the 0–1 variable x_{ij}^r satisfies $x_{ij}^r \in \{0, 1\}$ and $\sum_{r=1}^{\#H_{ij}} x_{ij}^r = 1$. Thus, model (12)–(14) and model (15)–(17) can be equivalently transformed into model (18)–(22) and model (23)–(27).

This completes the proof of Theorem 1. □

Note 1. Clearly, model (18)–(22) and model (23)–(27) are both mixed 0–1 linear programming models. Thus, if the Manhattan distance is used (i.e., Eq. (3)) to compute the CI of linguistic preference relations, the *ICI*(*H*) can be obtained by solving mixed 0–1 linear programming models.

Note 2. When the Euclidean distance is used (i.e., Eq. (4)) to compute the CI of linguistic preference relations, the interval consistency *ICI*(*H*) can be obtained by solving mixed 0–1 quadratic programming models. The analysis and results in the case of Euclidean distance are very similar to the Manhattan distance, for the sake of brevity, we have only presented the results based on the Manhattan distance in the rest of the paper.

3.2. Illustrative examples

In this subsection, we provide two examples to illustrate the use of the ICI of HFLPRs.

Example 2. Consider the HFLPR as follows,

$$H=\begin{pmatrix}\{s_4\} & \{s_2,s_3,s_4\} & \{s_5,s_6\} & \{s_4\}\\ \{s_6,s_5,s_4\} & \{s_4\} & \{s_1,s_2,s_3\} & \{s_6,s_7\}\\ \{s_3,s_2\} & \{s_7,s_6,s_5\} & \{s_4\} & \{s_4,s_5\}\\ \{s_4\} & \{s_2,s_1\} & \{s_4,s_3\} & \{s_4\}\end{pmatrix}.$$

To obtain the interval index of consistency of H, the following model to obtain WCI of H is constructed:

$$\begin{cases}\min\ 1-\frac{1}{48}\sum_{i<j<k}^{4}\left|z_{ij}+z_{jk}-z_{ik}-4\right|\\ s.t.\ \ z_{ij}=\sum_{r=1}^{\#H_{ij}}(x_{ij}^{r}\times\Delta^{-1}(H_{ij}^{r})) \qquad i=1,2,3,4;\ j=i+1,\ldots,4\\ \qquad \sum_{r=1}^{\#H_{ij}}x_{ij}^{r}=1 \qquad i=1,2,3,4;\ j=i+1,\ldots,4\\ \qquad x_{ij}^{r}=0\ or\ 1 \qquad i=1,2,3,4;\ j=i+1,\ldots,4;\ r=1,\ldots,\#H_{ij}\end{cases} \tag{28}$$

Model (28) can be equivalently transformed into the following mixed 0–1 linear programming model,

$$\begin{cases}\min\ 1-\frac{1}{48}\sum_{i<j<k}^{4}\left|z_{ij}+z_{jk}-z_{ik}-4\right|\\ s.t.\ z_{12}=2x_{12}^{1}+3x_{12}^{2}+4x_{12}^{3};\ \ z_{13}=5x_{13}^{1}+6x_{13}^{2};\ \ z_{14}=4x_{14}^{1}\\ \qquad z_{23}=x_{23}^{1}+2x_{23}^{2}+3x_{23}^{3};\ \ z_{24}=6x_{24}^{1}+7x_{24}^{2};\ \ z_{34}=4x_{34}^{1}+5x_{34}^{2}\\ \qquad x_{12}^{1}+x_{12}^{2}+x_{12}^{3}=1;\ x_{13}^{1}+x_{13}^{2}=1;\ x_{14}^{1}=1\\ \qquad x_{23}^{1}+x_{23}^{2}+x_{23}^{3}=1;\ x_{24}^{1}+x_{24}^{2}=1;\ x_{34}^{1}+x_{34}^{2}=1\\ \qquad x_{ij}^{r}=0\ or\ 1 \qquad i=1,2,3,4;\ j=i+1,\ldots,4;\ r=1,\ldots,\#H_{ij}\end{cases} \tag{29}$$

Solving model (29), we can get $WCI(H)=CI(L_1)=0.667$, where

$$L_1=\begin{pmatrix}s_4 & s_2 & s_6 & s_4\\ s_6 & s_4 & s_1 & s_7\\ s_2 & s_7 & s_4 & s_4\\ s_4 & s_1 & s_4 & s_4\end{pmatrix}.$$

Similarly, the following model to obtain the BCI of H is constructed:

$$\begin{cases}\max\ 1-\frac{1}{48}\sum_{i<j<k}^{4}\left|z_{ij}+z_{jk}-z_{ik}-4\right|\\ s.t.\ \ z_{ij}=\sum_{r=1}^{\#H_{ij}}(x_{ij}^{r}\times\Delta^{-1}(H_{ij}^{r})) \qquad i=1,2,3,4;\ j=i+1,\ldots,4\\ \qquad \sum_{r=1}^{\#H_{ij}}x_{ij}^{r}=1 \qquad i=1,2,3,4;\ j=i+1,\ldots,4\\ \qquad x_{ij}^{r}=0\ or\ 1 \qquad i=1,2,3,4;\ j=i+1,\ldots,4;\ r=1,\ldots,\#H_{ij}\end{cases} \tag{30}$$

Model (30) can be equivalently transformed into the following mixed 0–1 linear programming model,

$$\begin{cases}\max\ 1-\frac{1}{48}\sum_{i<j<k}^{4}\left|z_{ij}+z_{jk}-z_{ik}-4\right|\\ s.t.\ z_{12}=2x_{12}^{1}+3x_{12}^{2}+4x_{12}^{3};\ \ z_{13}=5x_{13}^{1}+6x_{13}^{2};\ \ z_{14}=4x_{14}^{1}\\ \qquad z_{23}=x_{23}^{1}+2x_{23}^{2}+3x_{23}^{3};\ \ z_{24}=6x_{24}^{1}+7x_{24}^{2};\ \ z_{34}=4x_{34}^{1}+5x_{34}^{2}\\ \qquad x_{12}^{1}+x_{12}^{2}+x_{12}^{3}=1;\ x_{13}^{1}+x_{13}^{2}=1;\ x_{14}^{1}=1\\ \qquad x_{23}^{1}+x_{23}^{2}+x_{23}^{3}=1;\ x_{24}^{1}+x_{24}^{2}=1;\ x_{34}^{1}+x_{34}^{2}=1\\ \qquad x_{ij}^{r}=0\ or\ 1 \qquad i=1,2,3,4;\ j=i+1,\ldots,4;\ r=1,\ldots,\#H_{ij}\end{cases} \tag{31}$$

Solving model (31), we can get $BCI(H)=CI(L_2)=0.833$, where

$$L_2=\begin{pmatrix}s_4 & s_4 & s_5 & s_4\\ s_4 & s_4 & s_3 & s_6\\ s_3 & s_5 & s_4 & s_4\\ s_4 & s_2 & s_4 & s_4\end{pmatrix}.$$

Thus, the ICI of H is obtained, $ICI(H)=[0.667,0.833]$.

Example 3. Consider the HFLPR as follows,

$$H'=\begin{pmatrix}\{s_4\} & \{s_1,s_2\} & \{s_5,s_6\} & \{s_4,s_5\} & \{s_3,s_4\} & \{s_2,s_3\}\\ \{s_7,s_6\} & \{s_4\} & \{s_7,s_8\} & \{s_6,s_7\} & \{s_6\} & \{s_4,s_5\}\\ \{s_3,s_2\} & \{s_1,s_0\} & \{s_4\} & \{s_3,s_4\} & \{s_2,s_3\} & \{s_2\}\\ \{s_4,s_3\} & \{s_2,s_1\} & \{s_5,s_4\} & \{s_4\} & \{s_4,s_5\} & \{s_3,s_4\}\\ \{s_5,s_4\} & \{s_2\} & \{s_6,s_5\} & \{s_4,s_3\} & \{s_4\} & \{s_3\}\\ \{s_6,s_5\} & \{s_4,s_3\} & \{s_6\} & \{s_5,s_4\} & \{s_5\} & \{s_4\}\end{pmatrix}.$$

To obtain the interval index of the consistency of H', the following model to obtain WCI of H' is constructed:

$$\begin{cases} \min \ 1-\frac{1}{240}\sum_{i<j<k}^{6}\left|\sum_{r=1}^{\#H'_{ij}}x_{ij}^{r}\Delta^{-1}(H'^{r}_{ij})+\sum_{r=1}^{\#H'_{jk}}x_{jk}^{r}\Delta^{-1}(H'^{r}_{jk})-\sum_{r=1}^{\#H'_{ik}}x_{ik}^{r}\Delta^{-1}(H'^{r}_{ik})-4\right| \\ s.t.\ \sum_{r=1}^{\#H'_{ij}}x_{ij}^{r}=1 \qquad i=1,2,\ldots,6;\ j=i+1,\ldots,6 \\ \quad x_{ij}^{r}=0 \ or \ 1 \qquad i=1,2,\ldots,6;\ j=i+1,\ldots,6;\ r=1,\ldots,\#H'_{ij} \end{cases} \tag{32}$$

Model (32) can be equivalently transformed into the following mixed 0–1 linear programming model,

$$\begin{cases} \min \ 1-\frac{1}{240}\sum_{i<j<k}^{6}\left|\sum_{r=1}^{\#H'_{ij}}x_{ij}^{r}\Delta^{-1}(H'^{r}_{ij})+\sum_{r=1}^{\#H'_{jk}}x_{jk}^{r}\Delta^{-1}(H'^{r}_{jk})-\sum_{r=1}^{\#H'_{ik}}x_{ik}^{r}\Delta^{-1}(H'^{r}_{ik})-4\right| \\ s.t.\ x_{12}^{1}+x_{12}^{2}=1;x_{13}^{1}+x_{13}^{2}=1;x_{14}^{1}+x_{14}^{2}=1;x_{15}^{1}+x_{15}^{2}=1;x_{16}^{1}+x_{16}^{2}=1 \\ \quad x_{23}^{1}+x_{23}^{2}=1;x_{24}^{1}+x_{24}^{2}=1;x_{25}^{1}=1;x_{26}^{1}+x_{26}^{2}=1 \\ \quad x_{34}^{1}+x_{34}^{2}=1;x_{35}^{1}+x_{35}^{2}=1;x_{36}^{1}=1 \\ \quad x_{45}^{1}+x_{45}^{2}=1;x_{46}^{1}+x_{46}^{2}=1;x_{56}^{1}=1 \\ \quad x_{ij}^{r}=0 \ or \ 1 \qquad i=1,2,\ldots,6;\ j=i+1,\ldots,6;\ r=1,\ldots,\#H'_{ij} \end{cases} \tag{33}$$

Solving model (33), we can get $WCI(H')=CI(L'_1)=0.883$, where

$$L'_1=\begin{pmatrix} s_4 & s_1 & s_6 & s_5 & s_3 & s_2 \\ s_7 & s_4 & s_7 & s_7 & s_6 & s_4 \\ s_2 & s_1 & s_4 & s_4 & s_2 & s_2 \\ s_3 & s_1 & s_4 & s_4 & s_5 & s_4 \\ s_5 & s_2 & s_6 & s_3 & s_4 & s_3 \\ s_6 & s_4 & s_6 & s_4 & s_5 & s_4 \end{pmatrix}.$$

Similarly, the following model to obtain the BCI of H' is constructed:

$$\begin{cases} \max \ 1-\frac{1}{240}\sum_{i<j<k}^{6}\left|\sum_{r=1}^{\#H'_{ij}}x_{ij}^{r}\Delta^{-1}(H'^{r}_{ij})+\sum_{r=1}^{\#H'_{jk}}x_{jk}^{r}\Delta^{-1}(H'^{r}_{jk})-\sum_{r=1}^{\#H'_{ik}}x_{ik}^{r}\Delta^{-1}(H'^{r}_{ik})-4\right| \\ s.t.\ \sum_{r=1}^{\#H'_{ij}}x_{ij}^{r}=1 \qquad i=1,2,\ldots,6;\ j=i+1,\ldots,6 \\ \quad x_{ij}^{r}=0 \ or \ 1 \qquad i=1,2,\ldots,6;\ j=i+1,.,6;\ r=1,\ldots,\#H'_{ij} \end{cases} \tag{34}$$

Model (34) can be equivalently transformed into the following mixed 0–1 linear programming model,

$$\begin{cases} \max \ 1-\frac{1}{240}\sum_{i<j<k}^{6}\left|\sum_{r=1}^{\#H'_{ij}}x_{ij}^{r}\Delta^{-1}(H'^{r}_{ij})+\sum_{r=1}^{\#H'_{jk}}x_{jk}^{r}\Delta^{-1}(H'^{r}_{jk})-\sum_{r=1}^{\#H'_{ik}}x_{ik}^{r}\Delta^{-1}(H'^{r}_{ik})-4\right| \\ s.t.\ x_{12}^{1}+x_{12}^{2}=1;x_{13}^{1}+x_{13}^{2}=1;x_{14}^{1}+x_{14}^{2}=1;x_{15}^{1}+x_{15}^{2}=1;x_{16}^{1}+x_{16}^{2}=1 \\ \quad x_{23}^{1}+x_{23}^{2}=1;x_{24}^{1}+x_{24}^{2}=1;x_{25}^{1}=1;x_{26}^{1}+x_{26}^{2}=1 \\ \quad x_{34}^{1}+x_{34}^{2}=1;x_{35}^{1}+x_{35}^{2}=1;x_{36}^{1}=1 \\ \quad x_{45}^{1}+x_{45}^{2}=1;x_{46}^{1}+x_{46}^{2}=1;x_{56}^{1}=1 \\ \quad x_{ij}^{r}=0 \ or \ 1 \qquad i=1,2,\ldots,6;\ j=i+1,\ldots,6;\ r=1,\ldots,\#H'_{ij} \end{cases} \tag{35}$$

Solving model (35), we can get $BCI(H')=CI(L'_2)=1$, where

$$L'_2=\begin{pmatrix} s_4 & s_1 & s_5 & s_4 & s_4 & s_3 \\ s_6 & s_4 & s_7 & s_6 & s_6 & s_5 \\ s_3 & s_1 & s_4 & s_3 & s_3 & s_2 \\ s_4 & s_2 & s_5 & s_4 & s_4 & s_3 \\ s_4 & s_2 & s_5 & s_4 & s_4 & s_3 \\ s_5 & s_3 & s_6 & s_5 & s_5 & s_4 \end{pmatrix}.$$

Thus, the ICI of H' is obtained, $ICI(H')=[0.883, 1]$.

4. Interval consistency measure vs. normalization method: a comparative study

In this section, we have made a comparative study between the proposed interval consistency measure and the normalization method [39].

4.1. Illustrate the essence of the normalization method

Here, we use the HFLPR H provided in Example 2 to illustrate how to obtain the NCI via the normalization method in Section 2.3.

In order to obtain the normalized HFLPR H^N in Example 2, the linguistic terms should be added to H_{13}, H_{24} and H_{34}. According to Definitions 7 and 8, we construct the normalized HFLPR $H^N(i<j)$ as follows,

$H_{ij}^N = H_{ij}$ for $(i, j) \neq (1, 3), (2, 4), (3, 4)$;
$H_{13}^N = \{s_5, (1-\varsigma) \times s_5 + \varsigma \times s_6, s_6\}$;
$H_{24}^N = \{s_6, (1-\varsigma) \times s_6 + \varsigma \times s_7, s_7\}$;
$H_{34}^N = \{s_4, (1-\varsigma) \times s_4 + \varsigma \times s_5, s_5\}$.
According to Eqs. (6) and (7),

$$NCI(H) = 1 - \frac{1}{48} \times \frac{1}{3} \sum_{\rho=1}^{3} \sum_{i<j<k}^{4} |\Delta^{-1}(H_{ij}^{N,\rho}) + \Delta^{-1}(H_{jk}^{N,\rho}) - \Delta^{-1}(H_{ik}^{N,\rho}) - 4| = \frac{CI(L_1^N) + CI(L_2^N) + CI(L_3^N)}{3} \tag{36}$$

where

$$L_1^N = \begin{pmatrix} s_4 & s_2 & s_5 & s_4 \\ s_6 & s_4 & s_1 & s_6 \\ s_3 & s_7 & s_4 & s_4 \\ s_4 & s_2 & s_4 & s_4 \end{pmatrix}$$

$$L_2^N = \begin{pmatrix} s_4 & s_3 & (1-\varsigma) \times s_5 + \varsigma \times s_6 & s_4 \\ s_5 & s_4 & s_2 & (1-\varsigma) \times s_6 + \varsigma \times s_7 \\ (1-\varsigma) \times s_3 + \varsigma \times s_2 & s_6 & s_4 & (1-\varsigma) \times s_4 + \varsigma \times s_5 \\ s_4 & (1-\varsigma) \times s_2 + \varsigma \times s_1 & (1-\varsigma) \times s_4 + \varsigma \times s_3 & s_4 \end{pmatrix}$$

$$L_3^N = \begin{pmatrix} s_4 & s_4 & s_6 & s_4 \\ s_4 & s_4 & s_3 & s_7 \\ s_2 & s_5 & s_4 & s_5 \\ s_4 & s_1 & s_3 & s_4 \end{pmatrix}.$$

According to Section 2.3, in order to obtain the optimized NCI of H and the optimized parameter $\varsigma (0 \le \varsigma \le 1)$, the following model is constructed.

$$\begin{cases} \max \quad NCI(H) \\ s.t. \ NCI(H) = \frac{CI(L_1^N)+CI(L_2^N)+CI(L_3^N)}{3} \\ \quad 0 \le \varsigma \le 1 \end{cases} \tag{37}$$

We apply model (37) to obtain $NCI(H) = 0.764$ with $\varsigma = 0$. Besides, we also obtain the normalized HFLPR, H^N, as follows,

$$H^N = \begin{pmatrix} \{s_4\} & \{s_2, s_3, s_4\} & \{s_5, s_5, s_6\} & \{s_4, s_4, s_4\} \\ \{s_6, s_5, s_4\} & \{s_4\} & \{s_1, s_2, s_3\} & \{s_6, s_6, s_7\} \\ \{s_3, s_3, s_2\} & \{s_7, s_6, s_5\} & \{s_4\} & \{s_4, s_4, s_5\} \\ \{s_4, s_4, s_4\} & \{s_2, s_2, s_1\} & \{s_4, s_4, s_3\} & \{s_4\} \end{pmatrix}.$$

From Eqs. (36) and (37), we conclude that the essence of the NCI of the HFLPR H is the average of the consistency index of the three linguistic preference relations L_1^N, L_2^N and L_3^N.

4.2. Connection among ICI, NCI and the average consistency measure

As mentioned in Section 4.1, the NCI is determined by the average of the consistency of several linguistic preference relations associated with a HFLPR. Naturally, the connection between the NCI and the ACI of HFLPRs needs to be studied. First, we propose a method to measure the ACI of a HFLPR as Definition 12 and Algorithm 1.

Algorithm 1: The procedure to obtain the ACI of HFLPRs.

---

1. Input the HFLPR H.
2. For each linguistic preference relation associated with H, L^z $(z = 1, 2, ..., \#N_H)$.
do
calculate the consistency degree of L^z,
$CI(L^z) = 1 - \frac{2}{3gn(n-1)(n-2)} \sum_{i,j,k=1}^{n} |\Delta^{-1}(l_{ij}^z) + \Delta^{-1}(l_{jk}^z) - \Delta^{-1}(l_{ik}^z) - \frac{g}{2}|$
End for
3. Calculate the average consistency degree of H,
$ACI(H) = \frac{1}{\#N_H} \times \sum_{z=1}^{\#N_H} CI(L^z)$
4. Output $ACI(H)$.

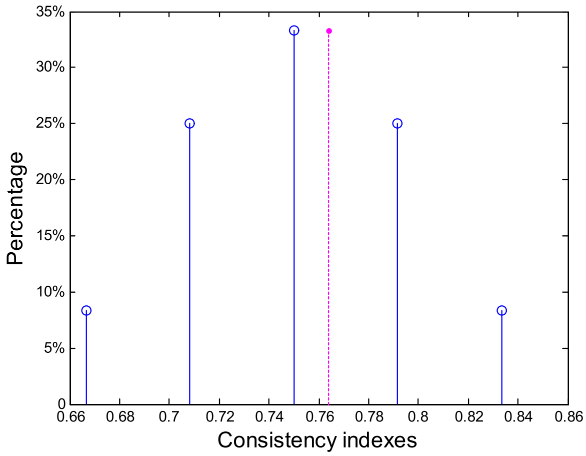

Fig. 1. The consistency distribution of all LPRs associated with H^1.

Definition 12. Let H be a HFLPR and let N_H be the set of all linguistic preference relations associated with H. The value of $ACI(H)$ is determined as the average consistency of all linguistic preference relations associated with the HFLPR, i.e.,

$$ACI(H) = \frac{1}{\#N_H} \times \sum_{L \in N_H} CI(L) \tag{38}$$

where $\#N_H$ is the number of linguistic preference relations in N_H, i.e., $\#N_H = \prod_{i=1}^{n} \prod_{j=i+1}^{n} (\#H_{ij})$.

It is clear that $ACI(H) \in [WCI(H), BCI(H)]$. Based on Definition 12, Algorithm 1 is provided to describe the procedure used to get the ACI.

To analyze the connection between ICI, NCI and ACI, we provide Example 4 as follows.

Example 4. Let $H^1 = H$ and $H^2 = H'$ be the HFLPRs provided in Examples 2 and 3. The HFLPRs H^3, H^4, H^5 and H^6 are from [17,34,39], respectively.

$$H^3 = \begin{pmatrix} \{s_4\} & \{s_5\} & \{s_6, s_7\} & \{s_5\} \\ \{s_3\} & \{s_4\} & \{s_0, s_1\} & \{s_6\} \\ \{s_2, s_1\} & \{s_8, s_7\} & \{s_4\} & \{s_1, s_2\} \\ \{s_3\} & \{s_2\} & \{s_7, s_6\} & \{s_4\} \end{pmatrix} \quad H^4 = \begin{pmatrix} \{s_4\} & \{s_3, s_4\} & \{s_5, s_6\} & \{s_5\} \\ \{s_5, s_4\} & \{s_4\} & \{s_7\} & \{s_3\} \\ \{s_3, s_2\} & \{s_1\} & \{s_4\} & \{s_0, s_1\} \\ \{s_3\} & \{s_5\} & \{s_8, s_7\} & \{s_4\} \end{pmatrix}$$

$$H^5 = \begin{pmatrix} \{s_4\} & \{s_3, s_4\} & \{s_5, s_6\} & \{s_1, s_2\} & \{s_0, s_1\} \\ \{s_5, s_4\} & \{s_4\} & \{s_6, s_7\} & \{s_2, s_3\} & \{s_0, s_1, s_2\} \\ \{s_3, s_2\} & \{s_2, s_1\} & \{s_4\} & \{s_5, s_6\} & \{s_4, s_5, s_6\} \\ \{s_7, s_6\} & \{s_6, s_5\} & \{s_3, s_2\} & \{s_4\} & \{s_4, s_5, s_6\} \\ \{s_8, s_7\} & \{s_8, s_7, s_6\} & \{s_4, s_3, s_2\} & \{s_4, s_3, s_2\} & \{s_4\} \end{pmatrix}$$

$$H^6 = \begin{pmatrix} \{s_4\} & \{s_1, s_2\} & \{s_6, s_7\} & \{s_1, s_2\} & \{s_4, s_5, s_6\} \\ \{s_7, s_6\} & \{s_4\} & \{s_4, s_5, s_6\} & \{s_1, s_2\} & \{s_0, s_1, s_2\} \\ \{s_2, s_1\} & \{s_4, s_3, s_2\} & \{s_4\} & \{s_5, s_6, s_7\} & \{s_3, s_4\} \\ \{s_7, s_6\} & \{s_7, s_6\} & \{s_3, s_2, s_1\} & \{s_4\} & \{s_3, s_4\} \\ \{s_4, s_3, s_2\} & \{s_8, s_7, s_6\} & \{s_5, s_4\} & \{s_5, s_4\} & \{s_4\} \end{pmatrix}$$

In order to demonstrate the connection between ICI and NCI, a simulation experiment has been made on the consistency distribution of all linguistic preference relations associated with the HFLPRs (see Figs. 1–6).

In Figs. 1–6, the x-axis shows the consistency indexes of all the linguistic preference relations associated with the HFLPRs $H^1, H^2, \ldots, H^6$, respectively. The y-axis shows the percentage of the consistency indexes of the linguistic preference relations associated with $H^1, H^2, \ldots, H^6$ and the red dotted line shows the position of NCI of $H^1, H^2, \ldots, H^6$.

Furthermore, Table 1 shows the ICI, ACI and NCI values of $H^1, H^2, \ldots, H^6$.

According to Figs. 1– 6 and Table 1, we have made the following observations:

(1) The NCI values presented in Zhu and Xu [39] are in the interval of ICI.
(2) The NCI values are very close to those of ACI. This shows that the normalization method is used to approximately measure the average consistency of the HFLPRs $H^1, H^2, \ldots, H^6$.

Note 3. As a result of the above numerical analysis, we conclude that the values of ACI and NCI of HFLPRs are very close. The main reason for this observation is that in numerical analysis we find that the consistency indexes of the linguistic preference relations associated with the HFLPR have an approximate normal distribution, and the NCI of the HFLPR is determined

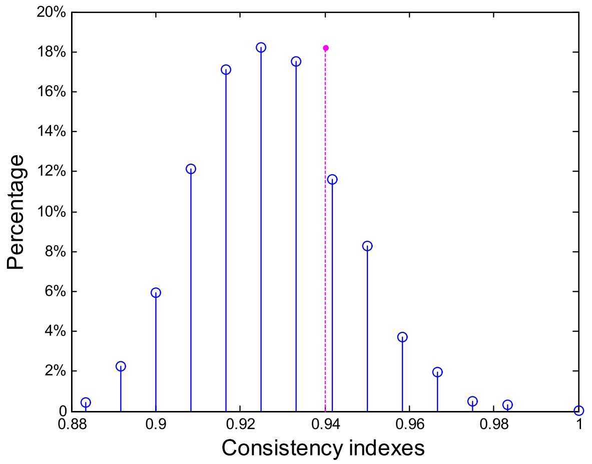

Fig. 2. The consistency distribution of all LPRs associated with H^2.

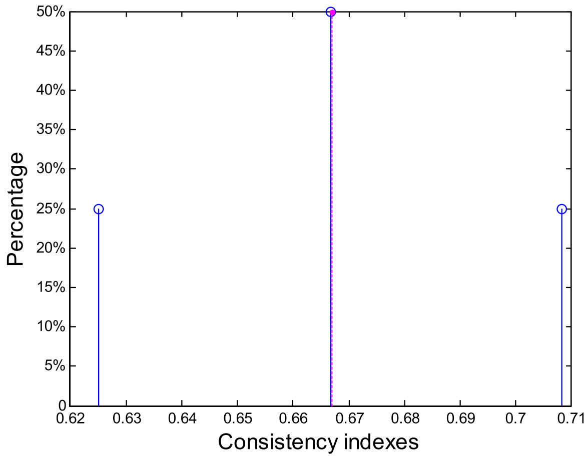

Fig. 3. The consistency distribution of all LPRs associated with H^3.

Table 1
The ICI, ACI and NCI of $H^1, H^2, \ldots, H^6$.

	H^1	H^2	H^3	H^4	H^5	H^6
ICI	[0.667, 0.833]	[0.883, 1]	[0.625, 0.708]	[0.833, 0.875]	[0.642, 0.842]	[0.592, 0.8]
ACI	0.75	0.927	0.667	0.854	0.734	0.692
NCI	0.764	0.94	0.667	0.854	0.75	0.694

by several linguistic preference relations whose consistency indexes are symmetrically distributed around ACI. Meanwhile, it should be noted that we cannot ensure that the values of the ACI and NCI of HFLPRs will be close in all cases due to the lack of analytical proof.

4.3. The difference between ICI and ACI (or NCI)

Here, we provide Example 5 to further analyze the difference between the ICI and the NCI, and show they behave differently when measuring the consistency degree of HFLPRs.

Example 5. Consider the following four HFLPRs,

$$H^7 = \begin{pmatrix} \{s_4\} & \{s_2, s_3, s_4\} & \{s_6, s_7, s_8\} & \{s_1, s_2, s_3\} \\ \{s_6, s_5, s_4\} & \{s_4\} & \{s_1, s_2, s_3\} & \{s_3, s_4, s_5\} \\ \{s_2, s_1, s_0\} & \{s_7, s_6, s_5\} & \{s_4\} & \{s_3\} \\ \{s_7, s_6, s_5\} & \{s_5, s_4, s_3\} & \{s_5\} & \{s_4\} \end{pmatrix}$$

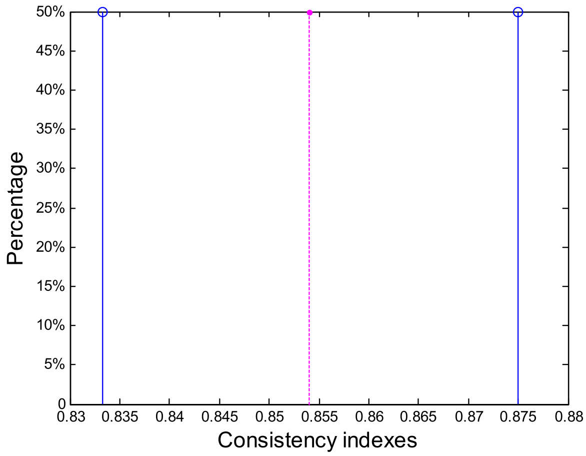

Fig. 4. The consistency distribution of all LPRs associated with H^4.

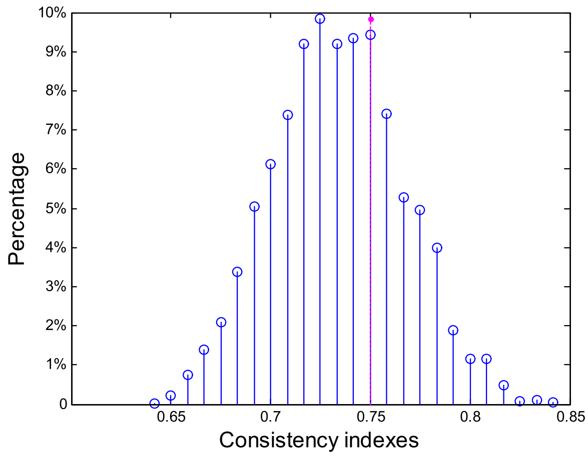

Fig. 5. The consistency distribution of all LPRs associated with H^5.

$$H^8 = \begin{pmatrix} \{s_4\} & \{s_3\} & \{s_6, s_7, s_8\} & \{s_7\} \\ \{s_5\} & \{s_4\} & \{s_1, s_2, s_3\} & \{s_3, s_4, s_5\} \\ \{s_2, s_1, s_0\} & \{s_7, s_6, s_5\} & \{s_4\} & \{s_3\} \\ \{s_1\} & \{s_5, s_4, s_3\} & \{s_5\} & \{s_4\} \end{pmatrix}$$

$$H^9 = \begin{pmatrix} \{s_4\} & \{s_6, s_7, s_8\} & \{s_4, s_5, s_6\} & \{s_5\} \\ \{s_2, s_1, s_0\} & \{s_4\} & \{s_4, s_5, s_6\} & \{s_3, s_4, s_5\} \\ \{s_4, s_3, s_2\} & \{s_4, s_3, s_2\} & \{s_4\} & \{s_1, s_2, s_3\} \\ \{s_3\} & \{s_5, s_4, s_3\} & \{s_7, s_6, s_5\} & \{s_4\} \end{pmatrix}$$

$$H^{10} = \begin{pmatrix} \{s_4\} & \{s_7\} & \{s_5\} & \{s_5\} \\ \{s_1\} & \{s_4\} & \{s_2, s_3, s_4\} & \{s_3, s_4, s_5\} \\ \{s_3\} & \{s_6, s_5, s_4\} & \{s_4\} & \{s_2\} \\ \{s_3\} & \{s_5, s_4, s_3\} & \{s_6\} & \{s_4\} \end{pmatrix}$$

Figs. 7 and 8 show the distribution of the consistency indexes of all the linguistic preference relations associated with H^7, H^8, H^9 and H^{10}, respectively, and the red dotted line shows the position of the NCI of HFLPRs. The ICI, ACI and NCI are included in Table 2.

According to Figs. 7 and 8 and Table 2, the following differences between ICI and ACI (or NCI) are highlighted:

(1) According to the ACI (or NCI) of H^7 and H^8, the consistency degrees of each are almost the same, while their ICI shows an obvious difference: The WCI of H^7 is much lower than the WCI of H^8 because $WCI(H^7) = 0.542$ and $WCI(H^8) = 0.667$, and the BCI of H^7 is much higher than the BCI of H^8 because $BCI(H^7) = 0.875$ and $BCI(H^8) = 0.75$.

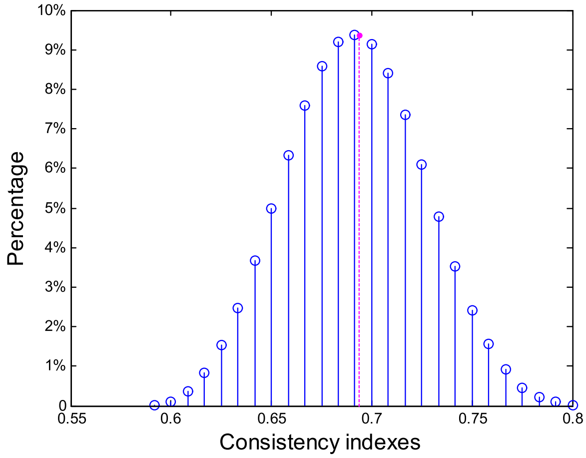

Fig. 6. The consistency distribution of all LPRs associated with H^6.

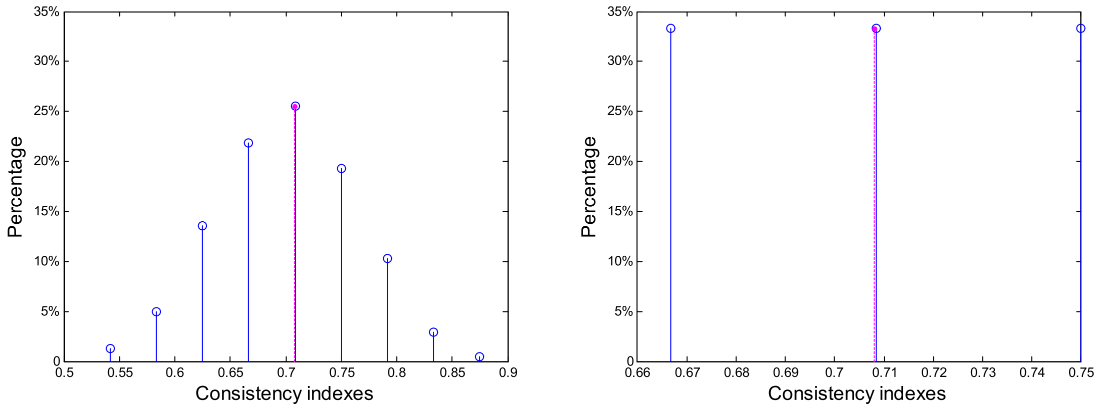

Fig. 7. The consistency distribution of all linguistic preference relations associated with the HFLPRs H^7 and H^8, respectively.

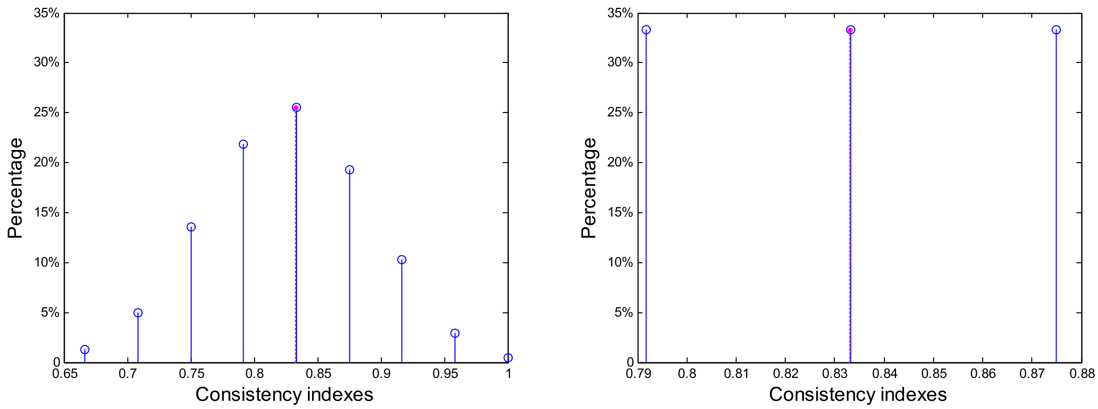

Fig. 8. The consistency distribution of all linguistic preference relations associated with the HFLPRs H^9 and H^{10}, respectively.

Table 2
The ICI, ACI and NCI of H^7, H^8, H^9 and H^{10}.

	H^7	H^8	H^9	H^{10}
ICI	[0.542, 0.875]	[0.667, 0.75]	[0.667, 1]	[0.792, 0.875]
ACI	0.702	0.708	0.826	0.833
NCI	0.708	0.708	0.833	0.833

(2) According to the ACI (or NCI) of H^9 and H^{10}, the consistency degree of each is almost the same, while their ICI shows an obvious difference: The WCI of H^9 is much lower than the WCI of H^{10} because $WCI(H^9) = 0.667$ and $WCI(H^{10}) = 0.792$, and the BCI of H^9 is much higher than the BCI of H^{10} because $BCI(H^9) = 1$ and $BCI(H^{10}) = 0.875$.

These observations show that the ACI (or NCI) and ICI have obviously different consistency reflection of HFLPRs. In the following, we further analyze the consistency indexes as they provide obviously different consistency degrees.

According to Eq. (38), the ACI of a HFLPR is determined by its associated linguistic preference relations, i.e.,

$$ACI(H) = \frac{1}{\#N_H} \times \sum_{L \in N_H} CI(L) \tag{39}$$

Similarly, based on Definition 9 and Eq. (36), the NCI of a HFLPR is determined by several linguistic preference relations associated with the HFLPR, have

$$NCI(H) = \frac{1}{c^N} \sum_{\rho=1}^{c^N} CI(H^{N,\rho}), \quad where\ H^{N,\rho} \in N_H. \tag{40}$$

Different from the ACI (or NCI), the ICI of a HFLPR is determined by its associated linguistic preference relation with the worst consistency degree and the best consistency degree. From Eqs. (9) and (10), we have

$$WCI(H) = \min_{L \in N_H} CI(L) \tag{41}$$

and

$$BCI(H) = \max_{L \in N_H} CI(L) \tag{42}$$

Obviously, it shows that $ACI(H) \approx NCI(H) \in ICI(H) = [WCI(H), BCI(H)]$.

In summary, the ICI provides the lower and upper bounds of the consistency of HFLPRs, and the ACI (or NCI) provides the average consistency degree of HFLPRs. Therefor, their combined use can better reflect the consistency status of HFLPRs.

5. Conclusion

People will often struggle to choose/hesitate when choosing between several linguistic terms when expressing their preferences, and the preference inconsistency is a popular issue in HFLPRs. This paper mainly focuses on the measurement of consistency of HFLPRs and the main contributions are as follows.

(1) We propose a new hesitant consistency measure, called interval consistency measure, to estimate the consistency range of a HFLPR.
(2) Generally, the normalization method is used as a tool to measure the consistency degree of a HFLPR, we show that the normalization method should be considered to be an approximate average consistency of a HFLPR via the detailed numerical analysis.

In the future, we plan to work on the following issues.

- The linguistic distribution is becoming a popular tool for modelling linguistic expressions with multiple linguistic terms in decision problems [38], it might be interesting to study ICI and ACI in preference relations with linguistic distributions.
- The consistency measurement has been used as a driver to improve the consistency degree [5,6], estimate the incomplete preference values [2,15] and set a numerical scale of linguistic term sets [4,9]. It might be interesting to study these issues in the decision making with HFLPRs based on the use of ICI and ACI as a driver.

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