An Efficient Power-Flow Approach Based on Heun and King-Werner's Methods for Solving Both Well and Ill-conditioned Cases

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Abstract – Solving the Power-Flow in ill-conditioned cases is still challenging as most of the available robust methodologies are not efficient enough to be widespread used in industry applications. This paper addresses this issue by developing a novel approach suitable for both ill and well-conditioned power cases. Since the developed approach arises from the combination of the King-Werner and Heun's methods, it is called Heun-King-Werner method. The developed approach naturally performs as a robust method in ill-conditioned cases and as a high order Newton-like method in well-conditioned systems, which makes it very suitable for solving both cases. The developed approach is tested using various realistic well and ill-conditioned cases under different demanding scenarios. Its performance is compared with other well-known Power-Flow methods. Results show that the developed approach is robust, reliable and computationally much more efficient than other well-known methods. In well-conditioned systems, it performs similar to the standard NR method but improving its convergence features in some cases. Based on the results, the developed approach may be widespread used in industry tools.

Keywords: Robust Power-flow calculation, well & ill-conditioned cases, Heun's method, King-Wener's Methods.

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Nomenclature

Abbreviation	
PF	Power-Flow
NR	Newton-Raphson method
HKW	Heun-King-Werner method
MLP	Maximum loadability point
RK4	4th order Runge-Kutta Power-Flow solver
RBS	Reverse-Bulirsch-Stoer approach
NR3	3rd order Newton-like method
Indexes and superscripts	
$i \in \mathbb{N}$	denotes the i^{th} bus of the system
$k \in \mathbb{N} \cup \{0\}$	denotes the k^{th} iteration of an iterative algorithm
$n_l \in \mathbb{N}$	total number of PQ buses
$n_g \in \mathbb{N}$	total number of PV buses
$n \in \mathbb{N}$	size of the PF ($n = 2n_l + n_g$ in polar coordinates)
$(\cdot)^{sch}$	scheduled value
Variables and constants	
$P \in \mathbb{R}$	nodal active power injection
$Q \in \mathbb{R}$	nodal reactive power injection
$V\angle\delta \in \mathbb{C}$	nodal voltage
$Y\angle\theta \in \mathbb{C}$	element of the admittance matrix
$\varepsilon \in \mathbb{R}^+$	convergence tolerance
$k_{\max} \in \mathbb{N}$	maximum number of iterations allowed
$h \in \mathbb{R}^+$	step size
$h_{\max} \in \mathbb{R}^+$	maximum step size allowed
$h_{\min} \in \mathbb{R}^+$	minimum step size allowed
$\mu \in \mathbb{R}^+$	parameter for initializing the step size
$\psi \in \mathbb{R}^+$	adaptation parameter used in the developed iterative procedure
$\alpha \in \mathbb{R}^+$	error threshold for updating the step size
$\tilde{\psi} \in \mathbb{R}^+$	threshold considered for switching to the standard NR
$\lambda \in \mathbb{R}^+$	loading level

Functions and operators	
$\mathbf{g}: \mathbb{R}^n \longmapsto \mathbb{R}^n$	PF equations
$[\cdot]^T: \mathbb{R}^{n\times m} \longmapsto \mathbb{R}^{m\times n}$	transpose operator
$[\cdot]^{-1}: \mathbb{R}^{n\times n} \longmapsto \mathbb{R}^{n\times n}$	inverse operator
$\lVert\cdot\rVert_\infty: \mathbb{R}^n \longmapsto \mathbb{R}$	Infinity norm
$\max\{a, b\}$	returns b if $b > a$ and a otherwise
$\min\{a, b\}$	returns b if $b < a$ and a otherwise
$SSR(\cdot): \mathbb{R}^n \longmapsto \mathbb{R}$	sum of squares of residuals

Matrices and vectors	
$\boldsymbol{x} \in \mathbb{R}^n$	PF state vector
$\boldsymbol{\delta_{PV}} \in \mathbb{R}^{n_g}$	voltage angles at PV buses
$\boldsymbol{\delta_{PQ}} \in \mathbb{R}^{n_l}$	voltage angles at PQ buses
$\boldsymbol{V_{PQ}} \in \mathbb{R}^{n_l}$	voltage magnitudes at PQ buses
$\mathbf{g}_{\boldsymbol{x}} \in \mathbb{R}^{n\times n}$	Jacobian matrix formed by the first partial derivatives of $\mathbf{g}$ with respect $\boldsymbol{x}$
$\boldsymbol{I} \in \mathbb{R}^{n\times n}$	Identity matrix

1. Introduction

In power system analysis, PF calculation plays a crucial role, where it is one of the most useful tools which finds multiple applications in planning, operation, optimization and control tasks among others [1]. Mathematically, PF is a nonlinear problem which must be solved using some numerical techniques. Although the solution of PF problem in polar coordinates using NR is the most common approach, a wide variety of different formulations and solvers have been proposed over decades. Interested readers can find a current literature review about this topic in [2] and references therein.

As any other nonlinear problem, PF cases might be well or ill-conditioned. Ill-conditioning of PF equations is mainly induced by large R/X ratios or high loading level [3]. While NR typically converges during few iterations in well-conditioned systems, convergence issues arise in ill-conditioned cases. These are strongly linked either with the singularity of Jacobian matrix in some points or with the local convergence character of the Newton's technique. Throughout this paper, a power system case will be categorized as ill-conditioned when, despite its solution exists, this solution is not reachable using NR and a flat start as initial guess [4].

In order to overcome the drawbacks showed by NR in case of solving ill-conditioned cases, a variety of reliable or robust approaches have been developed in the literature. The Iwamoto's method [5], is probably the most popular robust PF solution method, and typically used as benchmark (see e.g. [6, 7]). It is based of modifying the Newton's increment vector by calculating an optimal multiplier factor which is devoted for improving the convergence properties of iterative process. Although the Iwamoto's method is quite robust, it is frequently very slow since the optimal multiplier factor tends to be short as solution is approached [8]. Other multiple robust approaches based on the Iwamoto's method have been proposed over decades [9-11].

Based on the Continuous Newton's philosophy [12], any numerical integration method might be adapted for solving the PF problem. This idea was further investigated in [4]. In this reference, it was concluded that PF solution techniques based on the Continuous Newton's method constitute robust and efficient solvers. Numerical investigations have been carried out in a large realistic ill-conditioned case to prove the superiority with respect the Iwamoto's method. Nevertheless, these kinds of methodologies still present high computational burden, since as many Jacobian factorizations as order of the numerical technique employed are required each iteration. This issue has been addressed in [13], reducing the total number of Jacobian factorizations. Recently, a comparative study of different PF solution methods based on the Runge-Kutta formulas has been presented in [14].

A dynamic solution paradigm of PF problem has been developed in [15, 16]. Resulted algorithm is barely affected by the initial guess considered, showing widely convergence. Nevertheless, as any dynamic system, the PF has also to be solved using some integration routines. These schemes are very heavy, hence this paradigm is inappropriate for realistic large cases.

The Levenberg-Marquadt's method [17], is aimed to always find a solution of the PF equations, at least, in the sense of minimum squares. However, this solution does not always correspond to the real solution of the PF equations [17]. Recently, this methodology has been exploited for solving the PF problem of ill-conditioned cases [18, 19].

The holomorphic embedding method has recently been applied for solving the PF equations. In [20], it has been applied for solving the DC-PF and nonlinear DC circuits. In [21], several techniques based on holomorphic embedding method have been presented. These techniques are recursive rather than iterative, therefore, an infinite number of formulations exist, each one has different numerical properties. The computational performance of holomorphic embedding method has been tackled in [22], by introducing two novel methodologies.

Recently, two robust PF solution techniques based on the Bulirsch-Stoer method have been developed in [23]. They showed good performance for managing large-scale ill-conditioned cases, outperformed other methodologies.

Despite that most of power systems are naturally well-conditioned, ill-conditioned cases are becoming more frequent [16]. In addition, robust PF approaches are normally not suitable for industry applications. One can revise the results obtained with the Iwamoto's method in [4]. In the system studied here, this methodology often employs more than 1000 iterations, which is totally unaffordable for industry applications. On the other hand, methodologies studied in [4, 20, 23] present very high computational burden. The same occurs with the dynamic solution paradigm considered in [15, 16], which is totally not suitable for large-scale realistic systems. The main drawback of those methods based on the Levenberg-Marquardt technique is either the accuracy of the solution or the dependency of the parameters employed

[17]. In conclusion, the reader could deduce at this point that finding a PF technique with universal character is very difficult and it is not frequently addressed in the literature, where the works treated so far are typically focused on either well or ill-conditioned cases separately.

In order to properly address these issues, this paper aims to cover the following targets:

- Developing an efficient and robust PF solution approach which is suitable for industry tools. In that sense, it is aimed to develop a methodology more efficient than the other available robust techniques, but still reliable enough. On the other hand, its usage with respect to the most standard PF solution method (NR), should not be counterproductive. Since the developed approach arises from the combination of the King-Werner [24] and Heun's methods, it has been called Heun-King-Werner method. The developed approach naturally performs as a robust method for solving the ill-conditioned cases and as a high order Newton-like method in well-conditioned systems, which makes it very suitable for solving either well or ill-conditioned cases.
- The developed approach is numerically compared with the standard NR, and two Newton-like methods with high convergence rate [25, 26]. To do that, various realistic well and ill-conditioned systems are considered under different conditions.

Remainder of this paper is organized as follows. The developed PF solution approach is deeply explained in Section 2. Solution procedure of the developed HKW is fully described in Section 3 with an illustrative example. In Section 4, multiple numerical experiments are carried out in order to validate the developed approach. Finally, the main conclusions are presented in Section 5.

2. Developed HKW approach for solving PF problem

Solving systems of nonlinear equations is one of the most difficult problems in mathematical sciences [27]. Fig. 1 depicts three typical solutions of nonlinear equations using NR method. For the sake of simplicity, a case with $n = 1$ is shown. In Fig. 1a, a well-conditioned case is solved using NR, which likely converges in few iterations. Oppositely, NR may show some convergence difficulties in ill-conditioned cases. In the situation described in Fig. 1b, NR is likely to fail since the Jacobian matrix is not continuously invertible. On the face of the problem sketched in Fig. 1c, despite that Jacobian is continuously invertible, stability of NR depends on the considered starting point.

The condition number is frequently considered for measuring the degree of system ill-conditioning [28]. As the condition number increases, the degree of ill-conditioning grows as well. In this case, Fig. 1b is limited to the extreme case which the Jacobian matrix is not invertible (singular). Under this condition, the condition number of the studied system is expected to be very high.

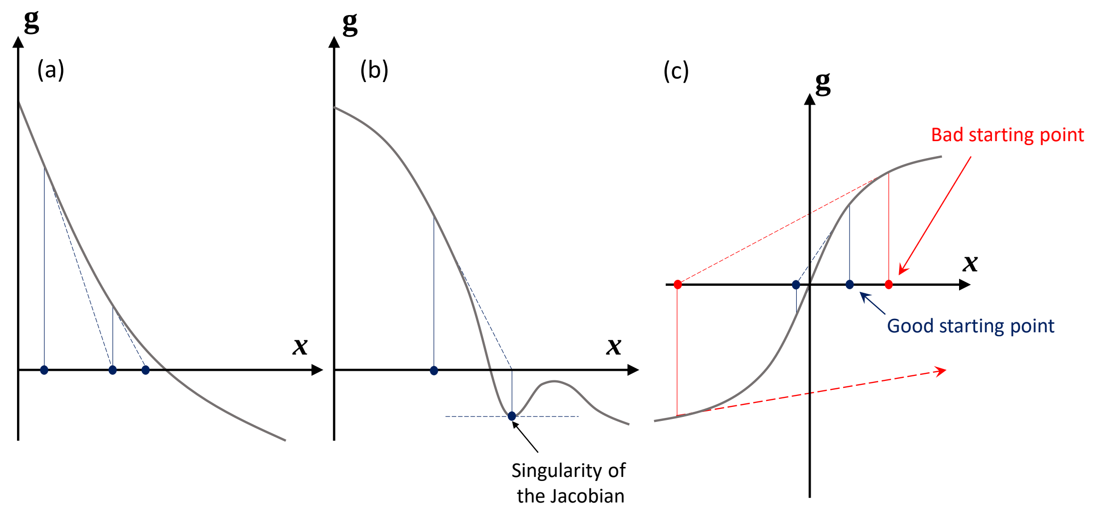

Fig. 1 Sketch of well (a) and ill-conditioned cases (b) and (c)

This paper is focused on developing a novel nonlinear approach robust enough to manage ill-conditioned cases. Unlike to other well-known PF solution methods, the developed approach aims to be efficient enough to be widespread used in both well and ill-conditioned scenarios.

The developed approach arises from properly combining two numerical methods. On the one hand, the Heun's method (also known as trapezoidal rule [29]), is used as main loop. On the other hand, we take the advantage of the high-order family of Newton-like techniques known as King-Werner methods [24]. The developed approach is consequently called Heun-King-Werner method (HKW).

King-Werner is a family of nonlinear solvers that reaches a convergence order of $1+\sqrt{2}$, superior to NR. In this case, the methodology developed in [30] has been considered. However, the following steps describe in detail the application of the developed HKW for solving PF.

<u>Step 0-Initialization:</u> read input data. Determine the system admittance matrix. Initialize iteration counter $k=0$. Initialize parameters and select a starting point $\boldsymbol{x}^{(0)}$

<u>Step 1:</u> Determine and evaluate the PF Jacobian matrix at $\boldsymbol{x}^{(k)}$. Then, calculate (1). Since PF Jacobian matrix is generally not definite positive, LU decomposition has to be used.

$$\Delta\boldsymbol{x}^{(k)} = -\left[\mathbf{g}_{\boldsymbol{x}}^{(k)}\right]^{-1}\mathbf{g}\left(\boldsymbol{x}^{(k)}\right) \qquad (1)$$

<u>Step 2-Euler's update:</u> as part of the Heun's method, state vector is initially updated using the forward Euler's technique as follows:

$$\boldsymbol{y}^{(k)} = \boldsymbol{x}^{(k)} + h^{(k)}\Delta\boldsymbol{x}^{(k)} \qquad (2)$$

This step equals to the Newton's increment vector, however, it is here modified by adding the step size. Thus, a high degree of robustness is expected since iterative procedure remains linear [4].

In this case, step size has to be initialized. To do that, we propose the following expression:

$$h^{(0)} = \max\left\{h_{\min}, \min\left\{h_{\max}, {}^{1}\!/_{\left(SSR^{(0)}\right)^{\mu}}\right\}\right\} \qquad (3)$$

where, SSR is given by:

$$SSR^{(k)} = \frac{1}{2}\left[\mathbf{g}\left(\boldsymbol{x}^{(k)}\right)\right]^T \mathbf{g}\left(\boldsymbol{x}^{(k)}\right) \quad (4)$$

Step 3-The King-Werner's update: as in [26], state vector is now updated by calculating the midpoint of the interval defined by $\boldsymbol{x}$ and $\boldsymbol{y}$ as follows:

$$\widetilde{\boldsymbol{x}}^{(k)} = \frac{1}{2}\left(\boldsymbol{x}^{(k)} + \boldsymbol{y}^{(k)}\right) \quad (5)$$

Step 4-Evaluation of the midpoint: in this step, PF equations are evaluated at the midpoint which calculated at previous step. Thus, an intermediate increment vector is calculated as follows:

$$\Delta\widetilde{\boldsymbol{x}}^{(k)} = -\left[\mathbf{g}_{\widetilde{\boldsymbol{x}}}^{(k)}\right]^{-1} \mathbf{g}\left(\widetilde{\boldsymbol{x}}^{(k)}\right) \quad (6)$$

Certainly, this step supposes an extra computational burden with respect to NR, since Jacobian matrix has to be evaluated and factorized twice each iteration. Nevertheless, this contributes outstanding features to the developed approach, as it is pointed out in following sections. Moreover, the impact on the overall computational performance of this step can be minimized (see Section 3.4).

Step 5-Heun's update: finally, state vector is updated using the Heun's method as follows:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \frac{h^{(k)}}{2}\left(\psi^{(k)}\Delta\boldsymbol{x}^{(k)} + \left(2 - \psi^{(k)}\right)\Delta\widetilde{\boldsymbol{x}}^{(k)}\right) \quad (7)$$

Parameter ψ is here introduced for naturally leading the developed approach to achieve quadratic convergence (see Appendix A). We recommend initializing this parameter $\psi^{(0)} = 1$, hence, Eq. (7) corresponds with the standard form of the Heun's method since both $\Delta\boldsymbol{x}$ and $\Delta\widetilde{\boldsymbol{x}}$ have the same weight. For following iterations, evolution of iterative procedure determines the value of this parameter (Step #7).

Step 6: if (8) is not satisfied, increase the iteration counter. If $k > k_{\max}$ or (8) is satisfied at $\boldsymbol{x}^{(k+1)}$ then, stop, otherwise go to Step 7.

$$\|\mathbf{g}(\boldsymbol{x})\|_\infty \leq \varepsilon \quad (8)$$

Step 7-Update parameters: the developed iterative procedure has two degrees of freedom namely h and ψ. These parameters are updated each iteration according to the following rules:

$$\epsilon = \left\| \boldsymbol{x}^{(k)} - \boldsymbol{y}^{(k-1)} \right\|_{\infty} \tag{9}$$

$$h^{(k)} = \begin{cases} \max\{0{,}9h, h_{\min}\} & \text{if } \epsilon > \alpha \\ \min\{1{,}1h, h_{\max}\} & \text{otherwise} \end{cases} \tag{10}$$

$$\psi^{(k)} = 2\left| \frac{SSR^{(k)} - SSR^{(0)}}{SSR^{(0)}} \right| \tag{11}$$

Once the parameters have been properly updated, return to Step #1.

In this case, the step size is adapted according to the error (9). A threshold $\alpha = 500$ has been considered in this paper. This threshold is difficult to be tuned *a priori*, since it is strongly case-dependent. Anyway, we have found this value suitable for studied cases and we honestly believe it works quite well in most problems, therefore, it can be considered universal or near universal. Regarding to $h_{\min}$ and $h_{\max}$, one should note that the algorithm described by steps 1-7 has quadratic convergence when $\psi = 2$ and $h = 1$ (a proof of that is provided in Appendix A). Hence, $h_{\max}$ should not ever be fixed greater than one. Regarding to $h_{\min}$, the shortest $h_{\min}$ implies the slowest convergence since the convergence of developed HKW becomes more linear. This strategy ensures a high degree of reliability. However, the developed algorithm may become too slow. Consequently, it can be tuned large and close to $h_{\max}$ in well-conditioned systems, on the other hand, it should be set as short as necessary while the degree of ill-conditioning grows. In this paper, $h_{\min} = 0.4$ and $h_{\max} = 1$ have been taken.

2.1.- Enhancing features of HKW approach

The developed HKW approach shows some interesting features, which are commented in the following subsections.

2.1.1.- Natural suitability of developed HKW in both well and ill-conditioned cases

Steps 1-7 described in Section 3, describe a naturally robust algorithm. In order to illustrate it, we refer to Fig. 2. In this figure, the process for solving an ill-conditioned case using HKW is sketched. In this case, scenario with $h = h_{\max} = 1$ is shown since it clearly supposes the most unfavorable situation. As it can be seen, naturally evolution of HKW leads it to reach the correct solution, while standard NR diverges.

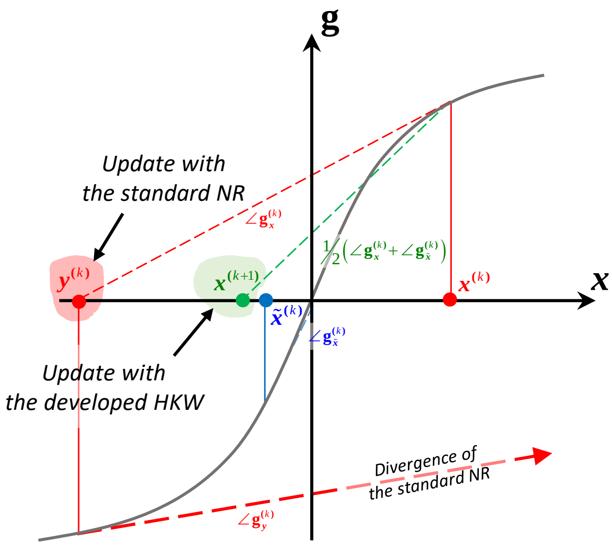

Fig. 2 Behavior of developed HKW in case of ill-conditioned cases

Since $h = 1$ in the analyzed scenario, the effect of the step size can be considered null. Therefore, only the developed iterative procedure is itself analyzed. Consequently, it can be concluded that the developed HKW constitutes a robust technique, independently of the value of the step size. This robust feature is undoubtedly due to its linear convergence characteristic when $\psi \neq 2$.

Now, let us analyze the behavior of HKW in well-conditioned systems. Fig. 3 sketches this scenario. As it can be seen, the evolution of HKW leads to reach the solution of the problem in less iterations than NR. This behavior is distinctive of high order Newton-like methods.

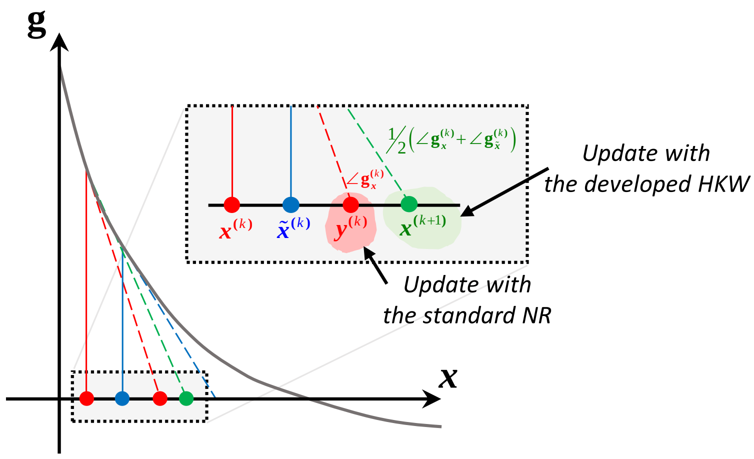

Fig. 3 Behavior of developed HKW in case of well-conditioned cases

Example sketched in Fig. 3 is intriguing. As we commented before, the developed HKW has linear convergence when $\psi \neq 2$, which supposes a priori slower convergence than NR. However, its linear convergence may be beneficious when the algorithm is evolving far away the solution. In this situation, NR frequently suffers slow convergence due to its quadratic characteristic is only ensured in the vicinity of the solution. The developed HKW is able to better approach the solution at first iterations. In other words, HKW performs better than NR far away the solution. This characteristic occasionally provides better convergence characteristics to HKW about traditional NR.

From previous examples, one can check that the developed HKW has an intrinsic adaptive character. This interesting feature makes the developed approach suitable for both well and ill-conditioned cases. This outstanding feature is difficult to find in most of available PF solution methods. On the one hand, robust techniques are normally not efficient enough, therefore, NR is normally preferred in well-conditioned systems. On the other hand, despite that high order Newton-like approaches typically enhance the convergence features of NR, high order Newton-like methods are not considered reliable due to its sensitive with respect to the initial guess [31].

2.1.2.- Natural evolution towards quadratic convergence

As we commented before, the developed HKW has quadratic convergence when $\psi = 2$ and $h = 1$ (see Appendix A). Adaptive mechanisms (10) and (11) have been developed so that both ψ and h, naturally evolve until achieve quadratic convergence characteristics (as NR).

2.2.- Multiple solutions

Due to the quadratic form of PF equations, they present two solutions or equilibrium points (except at MLP where the two solutions merge). These solutions are normally called high and low voltage due to their locations on the PV curve. Frequently, the high voltage solution is aimed to be calculated since it corresponds with the stable operating point of the system.

Reachability of each solution mainly depends on the starting point considered for initializing the iterative process. In Section 4, the developed HKW shows good ability for reaching the high voltage solution even starting from a flat start.

2.3.- Handling generators' reactive limits

Despite that the developed formulation does not directly take into account equipment's reactive limits, they might be easily incorporated using some available mechanisms for NR. One of the most common strategy [32, 33], proceeds as follows. Firstly, the PF is solved without considering the generators' reactive limits. When the solution is reached, it is checked if some reactive limit has been hit. In affirmative case, those PV buses which some reactive limits violated are converted to PQ buses. In these converted buses, the injected reactive power is replaced by the corresponding limit. Then, the PF is solved again. This process is repeated until a feasible solution is achieved or all system buses are converted to PQ type. In the latter case, the problem is declared unfeasible. Although the strategy explained above is considered the most standard one and it has been used in this work, it is worth to mention that the developed HKW is versatile enough for incorporating other strategies. For example, in [16] the generators' reactive limits are handled as inequality constraints. This mechanism might be adapted into the developed approach in the same way.

2.4.- Computational burden

It is well-known that the heaviest part of a PF solution procedure based on the Newton's method is the factorization of Jacobian matrix [4]. NR can be considered efficient enough since only needs to factorize the Jacobian matrix once each iteration. The developed HKW approach is, a priori, less efficient due to it needs two factorizations each iteration. However, as it has been explained in Section 3.1.1, the developed iterative procedure naturally reaches quadratic convergence when $\psi = 2$ and $h = 1$. In order to alleviate the computational burden of HKW approach, we propose to automatically switch the iterative process to the standard NR when ψ become greater than a predetermined threshold ($\tilde{\psi} \leq 2$), hence a factorization along with other vectors and matrix calculations are avoided.

For the sake of clarity, Fig. 4 shows the flowchart of developed HKW approach for solving PF problems in case of well and ill-conditioned cases. In addition, a pseudocode for the developed PF solution procedure is given in Appendix B. As it can be seen, this procedure doubles the computational burden of NR until $\psi > \tilde{\psi}$, at this point, computational effort is reduced until equal the computational burden of NR.

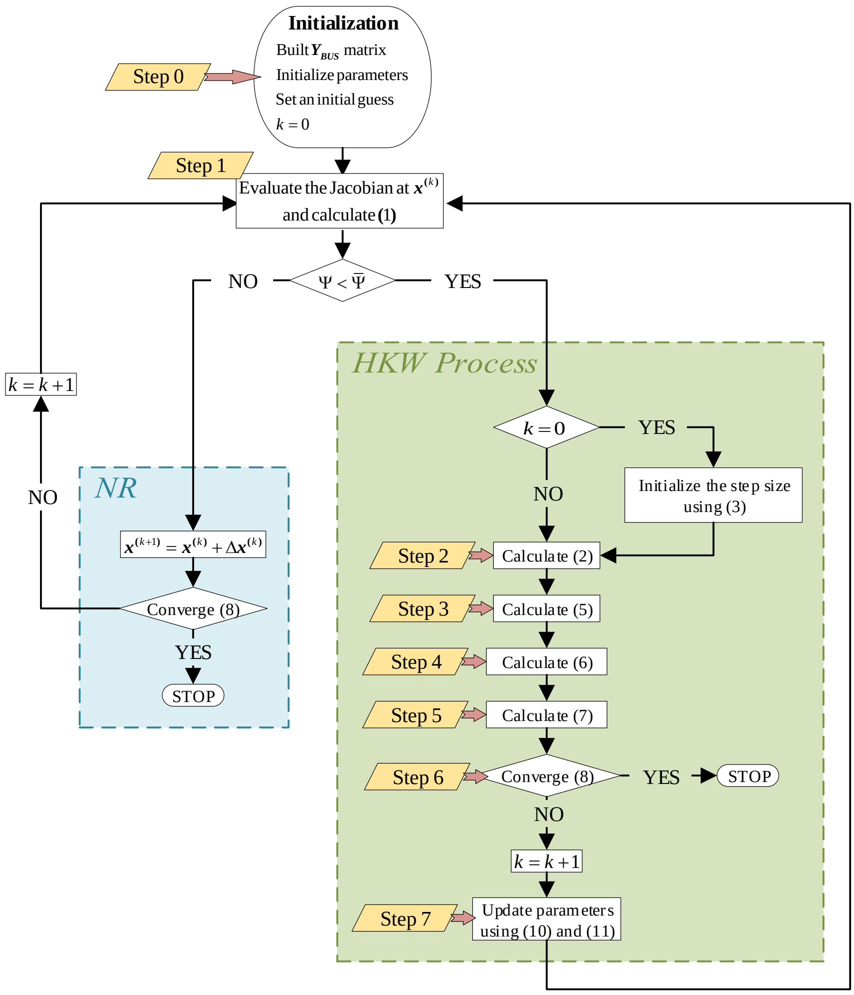

Fig. 4 Proposed flowchart of developed HKW approach for solving PF problems of well and ill-conditioned systems

Parameter $\tilde{\psi}$ should be tuned close enough to 2. Anyway, smaller $\bar{\psi}$ higher degree of efficiency since the developed algorithm switches to standard NR earlier. However, one should be careful since the reliability of developed approach might be affected if $\bar{\psi}$ is too small, especially when the algorithm is evolving far away the solution.

2.5.- Recommendations for tuning the involved parameters

- Parameter μ should be tuned as the step size is properly scaled as function of $SSR^{(0)}$. Considering that $SSR^{(0)}$ in the PF problem lies $\sim 10^4 - 10^5$, a value of $\mu = 0.06$ is considered suitable for PF purposes.

- When $\psi = 1$, both $\Delta\boldsymbol{x}$ and $\Delta\tilde{\boldsymbol{x}}$ have the same weight when the state vector is updated. This idea corresponds to the solution procedures sketched in Fig. 2 and 3, which are guessed to be robust in ill-conditioned cases and efficient in well-conditioned systems. Therefore, we recommend to set $\psi^{(0)} = 1$
- When $\psi = 2$, convergence features of HKW are quadratic. However, HKW adopts the same solution procedure that NR, which also has quadratic convergence, in order to avoid a LU decomposition. Parameter $\bar{\psi}$ is here introduced to check when the developed HKW is achieving quadratic convergence rate, so that, closer ψ to 2, more quadratic behaviour. Therefore, one can conclude that when ψ is close enough to 2, it is suitable to change to NR in order to alleviate the overall computational burden. Consequently, it is suitable to fix $\bar{\psi} \leq 2$. In ill-conditioned cases, it has been tuned $\bar{\psi} = 1.9$ in order to ensure the robustness of HKW, as it is shown in Section 4, NR fail in ill-conditioned situations, so that, it is counterproductive to switch to NR too early. This restriction can be relaxed in well-conditioned systems in order to achieve better computational performance.
- As it is shown in Appendix A, the developed HKW achieves quadratic convergence only at $h = 1$, otherwise, it remains linear which ensures a high degree of robustness. Consequently, we recommend to set $h_{\max} = 1$
- As it is shown in Appendix A, the developed HKW becomes linear when $h \neq 1$, so that, its convergence becomes slower. Thus, it is interesting to tune h as large as possible in order to keep the convergence features of HKW as quadratic as possible. Therefore, due to $h_{\min} = 0.4$ has turned out to be short enough to efficiently solve the PF in the studied systems using the developed HKW, there are not reasons for fixing it shorter. Anyway, it can be set as short as necessary.
- Parameter α is difficult to be tuned a priori. Our recommendation is tuning it quite large in order to avoid enlarge the step size too early. Thus, step size is only enlarged when

$\boldsymbol{x}^{(k)}$ and $\boldsymbol{y}^{(k-1)}$ are both in the same order. This normally occurs when the algorithm is properly converging. However, the authors have found that $\alpha = 500$ works quite well in both well and ill-conditioned cases.

3. An illustrative example

In order to fully explain the developed solution procedure described in Steps 1-7 of Section 2 and Fig. 4, let us consider the 2-bus example described in [16] and sketched in Fig. 5.

Table 1 Parameters used in simulations

Parameter	Value
$h_{\min}$	0.4
$h_{\max}$	1
μ	0.06
$\psi^{(0)}$	1
$\bar{\psi}$	1.9
α	500

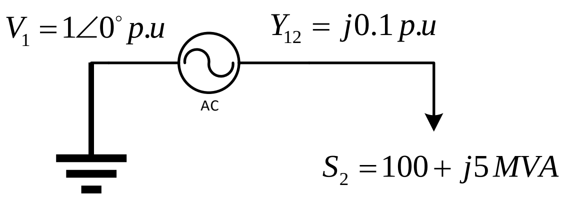

Fig 5 2-bus illustrative example [16]

Taking a flat start point, $\varepsilon = 10^{-5}$ and consider the parameters reported in Table 1, the developed solution procedure described in Fig. 4 solves the PF of system illustrated in Fig. 5 as follows:

<u>Step 0</u>

$$\boldsymbol{Y}_{\mathbf{BUS}} = \begin{bmatrix} -10j & 10j \\ 10j & -10j \end{bmatrix} \tag{12}$$

$$\boldsymbol{x}^{(0)} = [0,1]^T \tag{13}$$

<u>Step 1</u>

$$SSR^{(0)} = 0.063 \tag{14}$$

$$\mathbf{g}_{\boldsymbol{x}}^{(0)} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \tag{15}$$

$$\mathbf{g}\left(\boldsymbol{x}^{(0)}\right) = [0.1, 0.05]^T \tag{16}$$

$$\Delta\boldsymbol{x}^{(0)} = [-0.01, -0.005]^T \tag{17}$$

$$h^{(0)} = 1 \tag{18}$$

Step 2

$$\boldsymbol{y}^{(0)} = [-0.01, 0.995]^T \tag{19}$$

Step 3

$$\widetilde{\boldsymbol{x}}^{(0)} = [-0.005, 0.9975]^T \tag{20}$$

$$\mathbf{g}_{\widetilde{\boldsymbol{x}}}^{(0)} = \begin{bmatrix} 9.95 & -0.05 \\ -0.05 & 9.95 \end{bmatrix} \tag{21}$$

$$\mathbf{g}\left(\widetilde{\boldsymbol{x}}^{(0)}\right) = [0.05, 0.025]^T \tag{22}$$

Step 4

$$\Delta\widetilde{\boldsymbol{x}}^{(0)} = [-0.05, -0.04]^T \tag{23}$$

Step 5

$$\boldsymbol{x}^{(1)} = [-0.005, 0.0026]^T \tag{24}$$

Step 6

$$\mathbf{g}\left(\boldsymbol{x}^{(1)}\right) = [0.025, 0.013]^T \tag{25}$$

$$\left\|\mathbf{g}\left(\boldsymbol{x}^{(1)}\right)\right\|_{\infty} = 0.025 \tag{26}$$

Step 7

$$SSR^{(1)} = 3.95 \times 10^{-4} \tag{27}$$

$$\epsilon^{(1)} = 0.0028 \tag{28}$$

$$h^{(1)} = 1 \tag{29}$$

$$\psi^{(1)} = 1.87 \tag{30}$$

Remainder iterations are carried out in the same way and yields the following results. Note that $\psi^{(2)} = 1.99$, so that for $k \geq 3$ HKW proceeds as NR.

$$\left\|\mathbf{g}\left(\boldsymbol{x}^{(2)}\right)\right\|_{\infty} = 8.2 \times 10^{-4} \tag{31}$$

$$\left\|\mathbf{g}\left(\boldsymbol{x}^{(3)}\right)\right\|_{\infty} = 5.5 \times 10^{-8} \tag{32}$$

The solution is obtained for $k = 3$.

$$\boldsymbol{x}^{(3)} = [-0.01, 0.995]^T \tag{33}$$

4. Tests and results

With the aim of demonstrating that the developed HKW is versatile enough for handling both well and ill-conditioned cases, numerous experiments have been carried out. All simulations have been run under Windows 10 on a 3.4 GHz Intel Core i7-8750H CPU 2.2 GHz personal laptop (16.00 GB RAM). It is worth to mention that all reported solution times have been calculated as the average value of 100 simulations, with the aim of avoiding potential influence of parallel computational activities.

All considered PF techniques along the developed HKW have been coded in MATPOWER v7.0 [33]. In addition, the studied systems have been taken from the database of this software. We have considered realistic large-scale systems, so that they properly cover the main targets of this work. In all cases, $\varepsilon = 10^{-5}$ has been imposed as convergence criteria.

As it is commented in Section 2, the developed HKW uses various parameters. For tuning these parameters, guidelines provided in this section have been followed. Nevertheless, values of these parameters are facilitated in Table 1 for the sake of clarity. We cannot claim that the values reported in Table 1 are universal. However, these values should be used in most systems and practical applications since they are tuned based on logical criterion, and results obtained with them are satisfactory.

4.1.- Performance in ill-conditioned cases

Firstly, the developed HKW has been tested using the following ill-conditioned cases:

- *case3012wp*: 3012-bus snapshot of the Polish transmission system at winter 2007-08 evening peak [34].
- *case3375wp*: 3375-bus snapshot of the Polish transmission system at winter 2007-08 evening peak [34].
- *case3012wp (x2)*: duplicated version of the *case3012wp*, resulting in a system with 6024 buses. This case can be found in [35].
- *case3375wp (x2)*: duplicated version of the *case3012wp*, resulting in a system with 6750 buses. This case can be found in [35].
- *case13659pegase*: 13659-bus portion of European transmission system from PEGASE project [36, 37].
- *case13659pegase (x2)*: duplicated version of the *case13659pegase*, resulting in a system with 27318 buses. This case can be found in [35].

In addition, the main characteristics of studied ill-conditioned cases are collected in Table 2. The condition number of the studied cases have been also added in Table 2, this value is frequently considered for determining the degree of ill-conditioning. Comparing these values with those reported in reference [18], one can guess that studied cases can be categorized as ill-conditioned. Along with the developed solution procedure described in flowchart of Fig. 4 the following robust and standard PF techniques have been considered for comparison:

- Standard NR;
- Iwamoto's method [5];
- 4th order Runge-Kutta (RK4) [4];
- Heun's PF technique [14] ;

- High order Levenberg-Marquadt's (HLM) method developed in [18]. In this case, the so-called damping factor is updated using $\|\mathbf{g}\|_{\infty}^{1.3}$. Based on experience, this method achieves its best performance using this updating mechanism.
- Reverse Bulirsch-Stoer (RBS) [23];

Table 2 Studied ill-conditioned cases and their main characteristics

System	Buses	Branches	Generators	Load		n	Cond. Number
				MW	MVar		
case3012wp	3012	3572	502	27169	10200	5725	2.9×10^6
case3375wp	3374	4161	596	48363	19527	6355	4.3×10^6
case3012wp (x2)	6024	7145	770	54339	20401	11451	9.4×10^6
case3375wp (x2)	6748	8323	958	96726	39055	12711	3.5×10^7
case13659pegase	13659	20467	4092	381431	98523	23225	1.5×10^8
case31659pegase (x2)	27318	40935	8184	762863	197046	46451	1.6×10^8

Although NR is considered the most standard PF method, it fails to solve ill-conditioned cases according to the definition given in Section 1. It is included in the analysis just for verifying that the studied systems are naturally ill-conditioned. In all simulations of ill-conditioned cases, a flat start has been taken for initializing the iterative procedure.

In order to determine if the considered PF solution methods are accurate enough, solution found by the standard NR using the default starting point provided by MATPOWER is considered as the benchmark solution. Therefore, a solution is considered not accurate if the difference between it and the considered benchmark solution is greater than 0.1 pu in voltage magnitudes or 0.05 deg in voltage angles. In that sense, the low voltage value is also considered inaccurate.

4.1.1.- Base Load scenario

Firstly, the developed HKW is validated with the base load scenario of studied ill-conditioned cases. The total number of iterations required for the considered PF methods in this case are given in Table 3.

Table 3 Total number of iterations of different PF methods in case of ill-conditioned cases at base load scenario

System	NR	Iwam.	RK4	HLM	RBS	Heun	HKW
case3012wp	Diverge	2000	Diverge	14	13	31	7
case3375wp	Diverge	1218	Diverge	13	13	32	7
case3012wp(x2)	Diverge	1977**	Diverge	20	13**	32**	9
case3375wp(x2)	Diverge	Diverge	Diverge	23	15**	Diverge	9
case13659pegase	Diverge	43**	21	32	14	29**	7
case13659pegase(x2)	Diverge	41**	21	34	14	29**	9

*** Inaccurate solution*

As expected, NR diverged in all studied cases, which confirms that they are intrinsically ill-conditioned. The Iwamoto's method normally presented very slow convergence due to optimal multiplier tends to be very short as solution is approached. Moreover, it occasionally diverged or converged to an inaccurate solution (maybe the low voltage one). RK4 only successfully solved the *case13659pegase* and its duplicated version, after 21 iterations in both cases. RBS did not diverge, however, it reached an inaccurate solution in the *case3012wp(x2)* and *case3375wp(x2)*. On the other hand, the Heun's PF solution method only successfully solved the *case3012wp* and *case3375wp*, employing many iterations. HLM and the developed HKW converged to the high voltage (stable) solution in all studied systems. However, the developed HKW always employed less iterations than remainder methods.

Outstanding convergence characteristics of the developed HKW can be easily observed in Fig. 6. As it can be seen, the developed HKW gives the smallest residual after fifth iteration. Then, residual quickly decreases lower than 10^{-10}. This is also symptomatic of good robustness features.

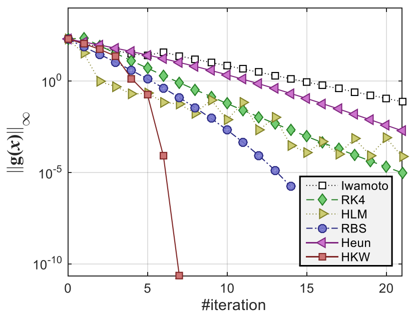

Fig. 6 Convergence characteristics in the *case13659pegase* at base load scenario

Table 4 reports the solution times of considered PF methods in the studied ill-conditioned cases at base load scenario.

Table 4 Solution times [s] of different PF solution methods in case of ill-conditioned cases at base load scenario

System	Iwam.	RK4	HLM	RBS	Heun	HKW
case3012wp	39.37	--	0.59	0.78	1.11	0.19
case3375wp	26.25	--	0.63	0.87	1.25	0.22
case3012wp(x2)	72.77**	--	1.69	1.53**	2.14**	0.47
case3375wp(x2)	--	--	2.22	2.06**	--	0.55
case13659pegase	3.92**	6.72	7.08	4.04	4.78**	0.89
case13659pegase(x2)	7.40**	14.35	15.26	8.27	9.87**	2.51

*** Inaccurate solution*

As expected, the Iwamoto's method was normally the slowest due to it employed a huge number of iterations to converge. Nevertheless, despite that it converged to an inaccurate solution, it achieved this solution in a reasonable number of iterations in the *case13659pegase* and its duplicated version. In these cases, it outperformed RK4, HLM, RBS and Heun's techniques. Anyway, the developed HKW notably outperformed the remainder considered PF techniques. One should note that RK4 and RBS present higher computational burden than the developed HKW. On the one hand, RK4 requires up to four factorizations each iteration. On the other hand, RBS starts factorizing the Jacobian matrix six times. HLM, Heun and the developed HKW show similar computational burden employing one or two LU decompositions each iteration (at least while $\psi < \bar{\psi}$), however, the developed HKW needed less number of iterations. It is worth to mention that a LU decomposition of HLM is much heavier due to the product $[\mathbf{g}_x]^T\mathbf{g}_x$, which produces a denser matrix.

4.1.2.- Limit load scenario

The impact of heavy loading level (λ) on the performance of developed HKW is also analyzed. To do that, we have considered the studied ill-conditioned cases at limit loading level, these cases can be found in [38], with the aim of allowing reproducibility on the results obtained. Table 5 presents the total number of iterations of different PF methods in this scenario.

Table 5 Total number of iterations of different PF solution methods in case of ill-conditioned cases at heavy load scenario

System	NR	Iwam.	RK4	HLM	RBS	Heun	HKW
case3012wp	Diverge	1847	Diverge	46	13	31	12
case3375wp	Diverge	1100	Diverge	47	13	32	12
case3012wp(x2)	Diverge	Diverge	Diverge	81**	Diverge	32**	15
case3375wp(x2)	Diverge	Diverge	Diverge	122	Diverge	49**	11
case13659pegase	Diverge	41	20**	163**	Diverge	29**	9
case13659pegase(x2)	Diverge	41	20**	168**	Diverge	29**	10

*** Inaccurate solution*

As expected, NR diverged in all studied cases. The Iwamoto's method diverged in the *case3012wp(x2)* and *case3375wp(x2)*. In the *case13659pegase* and its duplicated version, RK4 obtained an inaccurate solution and RBS diverged, oppositely, the Iwamoto's method successfully converged. HLM found inaccurate solutions in the *case3012wp(x2)*, *case31659pegase* and its duplicated version employing more iterations with respect to the base load scenario in the remainder cases, while the Heun's method just successfully solved the *case3012wp* and *case3375wp*. The developed HKW still successfully solved all considered cases, but more iterations were employed with respect the base load scenario. This is not disturbing since most PF techniques tend to employ more iterations when loading level is increased [26]. Moreover, the developed HKW still required less iterations than remainder PF techniques.

From Table 5, it can be guessed that the developed HKW is still the fastest technique in heavy loading scenario. Nevertheless, the solution times in this case have been reported in Table 6 for the sake of completeness. As expected, the developed HKW was faster than remainder PF techniques.

Table 6 Solution times [s] of different PF solution methods in case of ill-conditioned cases at limit load scenario

System	Iwam.	RK4	HLM	RBS	Heun	HKW
case3012wp	35.95	--	1.95	0.78	1.09	0.32
case3375wp	23.76	--	2.26	0.87	1.25	0.36
case3012wp(x2)	--	--	6.88**	--	2.14**	0.74
case3375wp(x2)	--	--	12.37	--	3.68**	0.70
case13659pegase	3.65	6.41**	36.10**	--	4.81**	1.11
case13659pegase(x2)	7.40	13.61**	75.53**	--	9.85**	2.73

*** Inaccurate solution*

4.1.3.- Reactive limits enforcement

Exploring the performance of developed HKW when the generators' reactive limits are taken into account, is interesting from the numerical point of view. If the strategy described in Section 2.3 is implemented, one should note that the size of system grows from a PF solution to the following one. This is due to, between two consecutives PF solutions, some PV buses are converted to PQ, consequently, each bus converted contributes with an extra variable. As result, this scenario demands outstanding requirements of PF techniques.

Table 7 reports the total number of iterations employed by different PF techniques when the generators' reactive limits are enabled. To do that, the strategy described in Section 2.3 has been implemented in all studied PF techniques. Number of iterations employed in this scenario mainly depends on the total number of PF solutions that a technique has to calculate for reaching a feasible solution. It is worth to notice that the results reported in Table 7 are just indicative. One should note that different number of PF calculations are normally required for achieving either the low or high voltage feasible solutions. In such cases, the average number of iterations (labelled AI) is considered a better indicator.

As expected, the developed HKW outperformed the remainder PF methods in both total and average number of iterations. Enhancing robustness of the developed HKW is still shown in this scenario, besides its ability to reach the high voltage solution even with a flat start.

Table 7 Comparison of different PF solution methods in case of ill-conditioned cases with generators' reactive limits enforcement

Method		*case3012wp*	*case3375wp*	*case3012wp (x2)*	*case3375wp (x2)*	*case13659pegase*	*case13659pegase (x2)*
NR		Diverge	Diverge	Diverge	Diverge	Diverge	Diverge
Iwam.	**TI**	4175	2811	8801**	Diverge	73**	Diverge
	PS	3	4	6**		2**	
	AI	1392	703	1467**		36.5**	
RK4	**TI**	Diverge	Diverge	Diverge	Diverge	34	34
	PS					2	2
	AI					17	17
HLM	**TI**	20	20	26	33	35	37
	PS	3	4	3	5	2	2
	AI	6.7	5	8.7	6.6	17.5	18.5
RBS	**TI**	29	32	62**	34**	24	24
	PS	3	4	6**	4**	2	2
	AI	9.7	8	10**	8.5**	12	12
Heun	**TI**	64	72	141**	Diverge	54**	Diverge
	PS	3	4	6**		2**	
	AI	21.3	18	23.5**		27**	
HKW	**TI**	13	15	15	19	10	12
	PS	3	4	3	5	2	2
	AI	4.3	3.8	5	3.8	5	6

*** Inaccurate solution*
TI: Total number of iterations for reaching a feasible solution
PS: Total number of PF solutions required for reaching a feasible solution
AI: Average number of iterations ($AI = TI/PS$)

On the light of results reported in Table 7, it is expected that the developed HKW is the fastest method in the considered scenario. This is confirmed from the solution times reported in Table 8. It is worth to mention that the reported average times (labelled AT), are always lower than those reported in Table 4. This is due to, between two consecutive PF calculations, the last reached solution is used for initializing the following PF problem, which always works better than a flat start initialization.

Table 8 Solution times [s] of different PF solution methods in case of ill-conditioned cases with generators' reactive limits enforcement

Method		*case3012wp*	*case3375wp*	*case3012wp (x2)*	*case3375wp (x2)*	*case13659pegase*	*case13659pegase (x2)*
NR		Diverge	Diverge	Diverge	Diverge	Diverge	Diverge
Iwam.	**TT**	83	61.9	333.5**	Diverge	6.44**	Diverge
	AT	27.7	15.5	55.6**		3.22**	
RK4	**TT**	Diverge	Diverge	Diverge	Diverge	10.79	22.78
	AT					5.39	11.39
HLM	**TT**	0.89	0.99	2.24	3.25	7.81	16.62
	AT	0.30	0.25	0.75	0.65	3.91	8.31
RBS	**TT**	1.85	2.53	7.73**	6.70**	6.84	14.02
	AT	0.62	0.63	1.29**	1.68**	3.42	7.01
Heun	**TT**	2.28	3.12	9.42**	Diverge	9.02**	Diverge
	AT	0.76	0.78	1.57**		4.51**	
HKW	**TT**	0.47	0.67	1.02	1.84	1.35	3.56
	AT	0.16	0.17	0.34	0.37	0.68	1.78

** Inaccurate solution*
TT: Total solution time [s] for reaching a feasible solution
AT: Average solution time [s] ($AT = TT/PS$), note that PS is reported in Table 7

4.1.4.- Failure scenarios

Now, let us consider some failure scenarios. These scenarios correspond with some stressing conditions of the system since some tie lines or generators are considered out of service. In this situation, the corresponding PV/QV curve of the system becomes narrower and, consequently, PF problem turns harder to solve [38].

In this Section, we have considered the *case3012wp* under the following scenarios:

- Fail 1: branches 9-11, 35-36 and 38-41 are opened. The load demanding at PQ and PV buses is increased by 26%.
- Fail 2: branch 9-11 is opened, generator connected with bus 24 is put out of service. The load demanding at PQ and PV buses is increased by 26%.
- Fail 3: branches 9-11 and 35-36 are opened, generators connected with bus 24 and 61 are put out of service. The load demanding at PQ and PV buses is increased by 25%.

Table 9 reports the total number of iterations required by different methodologies for solving the above failure scenarios.

Table 9 Total number of iterations required for solving the case3012wp at studied failure scenarios using different PF techniques

Failure	**NR**	**Iwam.**	**RK4**	**HLM**	**RBS**	**Heun**	**HKW**
#1	Diverge	1855	Diverge	39	13	31	9
#2	Diverge	1873	Diverge	39	13	31	9
#3	Diverge	1898	Diverge	37	13	31	9

As expected, NR and RK4 diverged in all scenarios. Remainder PF techniques successfully solved all considered scenarios. The developed HKW showed the highest convergence speed, requiring less iterations than the other PF methods. On the other hand, Table 10 reports the total execution time consumed by different methodologies in the failure scenarios considered. As expected, HKW was much faster than the remainder techniques.

Table 10 Solution times [s] of different PF techniques in case of the *case3012wp* at different failure scenarios

Failure	**NR**	**Iwam.**	**RK4**	**HLM**	**RBS**	**Heun**	**HKW**
#1	--	36.98	--	1.67	0.78	1.11	0.24
#2	--	37.14	--	1.67	0.78	1.11	0.24
#3	--	37.48	--	1.59	0.78	1.11	0.24

4.1.5.- Comparison with NR using different starting points

In previous scenarios, a flat start has been considered. This approach, although frequently considered in research works, is not widespread used in industry applications. Instead, an available snapshot is typically considered as starting point. If static state of the system does not vary abruptly (which is assumable in normal operational conditions), the current snapshot of the system is considered more reliable than a flat start.

With the aim to demonstrate that the developed HKW may outperforms NR even in industry applications, we have covered the scenario described above. To do that, the correct solution of the studied ill-conditioned cases have been obtained using the standard NR with the default starting point provided by MATPOWER. Then, taking this solution as mean of a Gaussian distribution, several starting points are generated using different standard deviations. These starting points have been used for initializing both NR and HKW iterative procedures. Fig. 7 shows the percentage of correct solutions achieved by each technique in both base and heavy load scenario. In this case, we have considered up to 100 different starting points for each standard deviation.

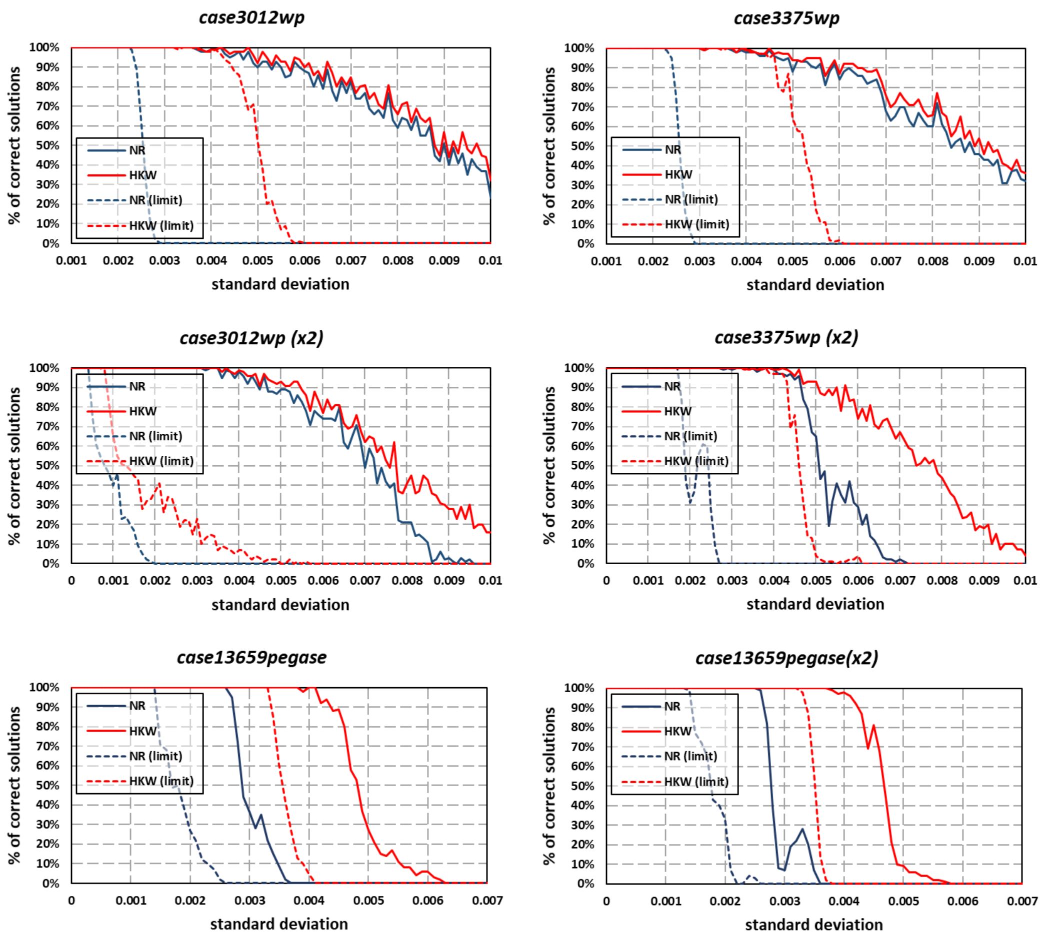

Fig. 7 Percentage of correct solutions achieved by NR and HKW using different starting points

The developed HKW is more reliable than NR, achieving more correct solutions in all cases. Superiority of developed approach is more noticeable at heavy load scenario. This demonstrates that, even in industry applications, the developed HKW might be offered higher degree of reliability than NR.

In fact, this analysis is an indirect way for calculating the Region of Attraction of developed HKW. Based on the obtained results, one can affirm that the Region of Attraction shown by HKW is notably wider than that produced by NR, consequently, the developed approach is more robust and reliable in badly-initialized systems.

4.2.- Performance in well-conditioned systems

Despite that the developed HKW is mainly envisaged for solving ill-conditioned cases, well-conditioned systems are more frequent in industry applications. With the aim to cover all possible scenarios, the performance of developed PF approach has been also tested in this kind of systems. For this purpose, several large-scale realistic cases taken from MATPOWER database are considered [34, 36, 37, 40].

Iterative process shown in Fig. 4 along with parameters values reported in Table 1, are suitable for ill-conditioned cases, however, HKW offers a high degree of versatility and different strategies can be implemented in order to obtain its optimal performance. In well-conditioned systems, some restricted conditions can be relaxed with the aim to obtain better results, thus, the following strategies have been considered:

- Strategy 1: HKW using the same iterative procedure shown in Fig. 4 and parameters reported in Table 1. This strategy has been used in previous scenarios.
- Strategy 2: HKW using a fixed step size mechanism. Thus, $h = 1$ has been taken constant during the iterative process. Updating mechanisms for the step size (3) and (10) are consequently omitted. Remainder steps and parameters are taken as in Strategy 1.
- Strategy 3: same procedure of Strategy 2 but taking $\bar{\psi} = 1{,}5$

While Strategy 1 is quite reliable and, consequently, suitable for ill-conditioned cases, it is expected that Strategies 2 and 3 show better performance in well-conditioned systems.

On the other hand, it was pointed out that the developed HKW performs as a high order Newton-like method in well-conditioned systems. In order to verify that, the high-order PF solution method (HNR) introduced in [26] along with the 3rd order Newton-like method (NR3) developed in [25] (which has already been considered for PF analysis in [41]), has been also included in the analysis along with the standard NR.

Total number of iterations and solution times in several well-conditioned systems are reported in Tables 11 and 12, respectively. In all cases, a flat start has been considered.

Table 11 Total number of iterations of different PF solution methods in case of well-conditioned systems

System	NR	HNR [26]	NR3 [25]	HKW Strategy: #1	#2	#3
case300	4	4	3	5	4	4
case1354pegase	4	4	3	6	4	4
case2869pegase	5	4	3	6	4	4
case2383wp	4	3	3	5	4	4
case2736sp	6	5	Diverge	6	5	5
case2737sop	5	4	4	6	5	4
case2746wop	6	5	4	7	5	5
case2746wp	6	5	4	7	5	5
case3120sp	5	4	4	6	5	5
case9241pegase	5	5	Diverge	7	5	5

Table 12 Solution times [s] of different PF solution methods in case of well-conditioned systems

System	NR	HNR [26]	NR3 [25]	HKW Strategy: #1	#2	#3
case300	0.01	0.01	0.01	0.02	0.01	0.01
case1354pegase	0.04	0.04	0.03	0.07	0.05	0.04
case2869pegase	0.10	0.09	0.07	0.18	0.12	0.10
case2383wp	0.06	0.05	0.05	0.11	0.09	0.08
case2736sp	0.11	0.10	--	0.16	0.11	0.11
case2737sop	0.09	0.10	0.08	0.16	0.13	0.10
case2746wop	0.10	0.09	0.10	0.18	0.13	0.11
case2746wp	0.10	0.09	0.07	0.17	0.12	0.11
case3120sp	0.10	0.09	0.09	0.18	0.15	0.13
case9241pegase	0.34	0.39	--	0.65	0.47	0.40

In all studied systems, the developed HKW, NR and HNR performed similarly. It is worth to mention that the developed HKW showed superior convergence features than standard NR when Strategy 2 or 3 have been implemented. This fact confirms that the developed technique naturally performs like a high order Newton-like method in well-conditioned systems.

Fig. 8 plots the convergence characteristics of different considered methods in the *case2736sp*. In this case, NR3 is not able to properly approach to the solution, and the residual grew during the iterative procedure, which has been considered a divergent case. As it was pointed out, high order Newton-like techniques are more sensitive to the starting point [31], which supposes the main obstacle for the widespread usage of these techniques in industry applications. This has been also observed in the *case9241pegase* which NR3 diverged. In such cases, the HKW supposed a reliable alternative to high order Newton-like techniques.

From Fig. 8, it can be also observed that the outstanding convergence characteristics of HKW. In this case, NR, HNR and HKW implemented with strategies 2 and 3 performed similar until second iteration. This is due to the solution is still far away and convergence features of these techniques remained linear. After that, convergence features of NR and HNR have been deteriorated. It means that the developed HKW with strategies 2 or 3 performs better than NR and HNR when the iterative procedure is carried out far away the solution. In this case, HKW with strategies 2 or 3 is able to better approximate the solution vector, which enables that convergence characteristics preserved, reaching the required convergence tolerance in less iterations than NR. This behavior can be also observed in the *case2869pegase*, *case2737sop*, *case2746wop* and *case2746wp*. Regarding to HKW with strategy 1, convergence remained linear due to the influence of step size until algorithm switched to NR.

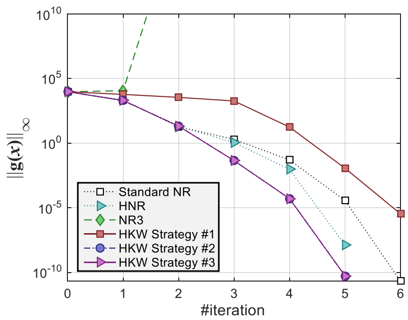

Fig. 8 Convergence characteristics in the *case2736sp*

As expected, the developed HKW was always slower than NR when strategy 1 is used due to its higher computational burden. However, the versatility of developed approach allows it reaches similar execution times of standard NR and NR3 implementing some accelerated strategy.

Comparing strategies 2 and 3, similar results have been obtained. However, strategy 3 is always faster due to algorithm switches to standard NR earlier, which is reflected in a lighter computational effort. Also, the convergence characteristics are slightly enhanced when strategy 3 is considered, due to the quadratic convergence features of NR are achieved earlier.

5. Conclusions

In this paper, a novel approach for solving the PF in well and ill-conditioned cases has been developed. It arises for the properly combination of the Heun's method and the King-Werner family of Newton-like techniques. Consequently, it has been called Heun-King-Werner. The developed approach shows some interesting features. Firstly, it naturally adopts a robust behavior in ill-conditioned cases, while it performs like a high order Newton-like method in well-conditioned systems, which makes it very suitable for solving either well or ill-conditioned scenarios. Secondly, structure of the developed algorithm based on the introduced HKW, naturally leads the iterative process towards a quadratic convergence characteristic. Due to this feature, we have proposed to switch to standard NR when a predetermined threshold is achieved in order to alleviate its computational burden. Overall, this is reflected in a reduced computational burden, especially if it is compared with other robust methodologies proposed in the literature.

In order to validate the developed PF approach, it has been tested in a variety of realistic ill-conditioned cases under different demanding conditions. In all studied experiments, the developed HKW showed an outstanding robust and efficient behavior, outperforming other well-known standard and robust PF solution techniques. While other studied PF solution techniques occasionally showed different convergence difficulties, the developed HKW successfully solved all studied cases.

Instead of a flat start, an available snapshot of the static operational of the system is typically used for initializing the PF solution procedure in industry applications. In order to fully explore the possibilities of the developed approach in industry tools, it was also compared with the standard NR in this common scenario. In this case, the developed HKW showed higher degree of reliability than NR.

For similar reasons, the developed approach has been also tested in a wide variety of realistic well-conditioned systems. In this numerical experiment, the performance of developed HKW has been compared with the standard NR along with two higher-order Newton-like methods [25, 26]. Versatility of developed approach has been also discussed, proposing three different strategies in order to get the best performance. Results of these systems have showed that the developed approach might improve the convergence features of standard NR and [26] if a suitable strategy is implemented. This demonstrates that the developed HKW naturally performs like a high order Newton-like method in well-conditioned systems. On the other hand, superior robustness with respect the high order Newton-like method [25] has been also proved.

Future works should be devoted to investigating the applicability of the developed HKW to other related problems like the Continuation Power-Flow [42].

Appendix A: Proof of convergence of the developed HKW

In this Appendix, we prove that the developed HKW has quadratic convergence when $\psi = 2$ and $h = 1$.

Theorem. *Let* $\mathbf{g}\colon \mathrm{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}^n$ *be sufficiently differentiable at each point of an open neighborhood D of* $\boldsymbol{r} \in \mathbb{R}^n$*, this is a solution of the system* $\mathbf{g}(\boldsymbol{x}) = \mathbf{0}$*. Let us suppose that* $\mathbf{g}(\boldsymbol{x})$ *is continuous and nonsingular in* $\boldsymbol{x}$*. Then the sequence* $\{\boldsymbol{x}^{(k)}\}_{k\geq 0}$ *obtained using the developed HKW converges to* $\boldsymbol{r}$ *with order 2 when* $\psi = 2$ *and* $h = 1$*. Otherwise, its convergence characteristic is linear.*

Proof. From Taylor expansion of $\mathbf{g}(\boldsymbol{x})$ and $\mathbf{g}_{\boldsymbol{x}}$ about $\boldsymbol{r}$ we obtain:

$$\mathbf{g}(\boldsymbol{x}) = \mathbf{g}_{\boldsymbol{x}}(\boldsymbol{r})[\boldsymbol{e} + \boldsymbol{C}_2\boldsymbol{e}^2 + \boldsymbol{C}_3\boldsymbol{e}^3] + \boldsymbol{O}(\boldsymbol{e}^4) \tag{36}$$

$$\mathbf{g}_{\boldsymbol{x}} = \mathbf{g}_{\boldsymbol{x}}(\boldsymbol{r})[\boldsymbol{I} + 2\boldsymbol{C}_2\boldsymbol{e} + 3\boldsymbol{C}_3\boldsymbol{e}^2] + \boldsymbol{O}(\boldsymbol{e}^3) \tag{37}$$

where $\boldsymbol{C}_j = (1/j!)[\mathbf{g}_{\boldsymbol{x}}(\boldsymbol{r})]^{-1}\mathbf{g}^{(j)}(\boldsymbol{r}) \in \mathbb{R}^{n\times n}$, $j = 2,3, \ldots$, and $\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{r} \in \mathbb{R}^n$

Inversion of $\mathbf{g}_{\boldsymbol{x}}$ gives (see [43]):

$$[\mathbf{g}_{\boldsymbol{x}}]^{-\mathbf{1}} = [\boldsymbol{I} - 2\boldsymbol{C}_2\boldsymbol{e} + (4\boldsymbol{C}_2 - 3\boldsymbol{C}_3)\boldsymbol{e}^2][\mathbf{g}_{\boldsymbol{x}}(\boldsymbol{r})]^{-\mathbf{1}} + \boldsymbol{O}(\boldsymbol{e}^3) \tag{38}$$

Thus we obtain:

$$\boldsymbol{d}_{\mathbf{1}} = [\mathbf{g}_{\boldsymbol{x}}]^{-\mathbf{1}}\mathbf{g}(\boldsymbol{x}) = \boldsymbol{e} - \boldsymbol{C}_2\boldsymbol{e}^2 + 2(\boldsymbol{C}_2^2 - \mathbf{3}\boldsymbol{C}_3)\boldsymbol{e}^3 + \boldsymbol{O}(\boldsymbol{e}^4) \tag{39}$$

From (2) and (5) we can write:

$$\boldsymbol{y} = \boldsymbol{x} - h[\mathbf{g}_{\boldsymbol{x}}]^{-\mathbf{1}}\mathbf{g}(\boldsymbol{x}) \Rightarrow \underbrace{\boldsymbol{y} - \boldsymbol{r}}_{\boldsymbol{e}_y} = \boldsymbol{x} - h[\mathbf{g}_{\boldsymbol{x}}]^{-\mathbf{1}}\mathbf{g}(\boldsymbol{x}) - \boldsymbol{r} = (1-h)\boldsymbol{e} + h\boldsymbol{C}_2\boldsymbol{e}^2 - 2h\left(\boldsymbol{C}_2^2 - \boldsymbol{C}_3\right)\boldsymbol{e}^3 + \boldsymbol{O}(\boldsymbol{e}^4) \tag{40}$$

$$\widetilde{\boldsymbol{x}} = \frac{1}{2}(\boldsymbol{x} + \boldsymbol{y}) \Rightarrow \underbrace{\widetilde{\boldsymbol{x}} - \boldsymbol{r}}_{\boldsymbol{e}_{\widetilde{x}}} = \frac{1}{2}(\boldsymbol{x} + \boldsymbol{y}) - \boldsymbol{r} = \frac{1}{2}\left(\boldsymbol{e} + \boldsymbol{e}_y\right) = \left(\frac{3}{2} - h\right)\boldsymbol{e} + h\boldsymbol{C}_2\boldsymbol{e}^2 - 2h\left(\boldsymbol{C}_2^2 - \boldsymbol{C}_3\right)\boldsymbol{e}^3 + \boldsymbol{O}(\boldsymbol{e}^4) \tag{41}$$

From Taylor expansion of (5) about $\boldsymbol{r}$ we obtain:

$$\mathbf{g}(\widetilde{\boldsymbol{x}}) = \mathbf{g}_{\boldsymbol{x}}(\boldsymbol{r})\left[\boldsymbol{e}_{\widetilde{x}} + \boldsymbol{C}_2\boldsymbol{e}_{\widetilde{x}}^2 + \boldsymbol{C}_3\boldsymbol{e}_{\widetilde{x}}^3\right] + \boldsymbol{O}(\boldsymbol{e}_{\widetilde{x}}^4) \tag{42}$$

$$[\mathbf{g}_{\widetilde{\boldsymbol{x}}}]^{-\mathbf{1}} = [\boldsymbol{I} - 2\boldsymbol{C}_2\boldsymbol{e}_{\widetilde{x}} + (4\boldsymbol{C}_2 - 3\boldsymbol{C}_3)\boldsymbol{e}_{\widetilde{x}}^2][\mathbf{g}_{\boldsymbol{x}}(\boldsymbol{r})]^{-\mathbf{1}} + \boldsymbol{O}\left(\boldsymbol{e}_{\widetilde{x}}^3\right) \tag{43}$$

At this point, we can calculate:

$$\boldsymbol{d}_{\mathbf{2}} = [\mathbf{g}_{\widetilde{\boldsymbol{x}}}]^{-\mathbf{1}}\mathbf{g}(\widetilde{\boldsymbol{x}}) = \left(\frac{3}{2} - h\right)\boldsymbol{e} - \boldsymbol{C}_2\left(\frac{3}{2} - 2h\right)\boldsymbol{e}^2 + \boldsymbol{O}(\boldsymbol{e}^3) \tag{44}$$

$$\boldsymbol{d}_{\mathbf{1}} + \boldsymbol{d}_{\mathbf{2}} = \left(\frac{5}{2} - h\right)\boldsymbol{e} - \boldsymbol{C}_2\left(\frac{5}{2} - 2h\right)\boldsymbol{e}^{\mathbf{2}} + \boldsymbol{O}(\boldsymbol{e}^3) \tag{45}$$

And the function vector at $k+1$ can be written using (7) as follows.

$$\boldsymbol{e}^{(k+1)} = \boldsymbol{e}^{(k)} - \frac{1}{2}h^{(k)}\left[\psi^{(k)}\boldsymbol{d}_{\mathbf{1}}^{(k)} + \left(2 - \psi^{(k)}\right)\boldsymbol{d}_{\mathbf{2}}^{(k)}\right] + \boldsymbol{O}\left(\boldsymbol{e}^{(k)^3}\right) \tag{46}$$

After some manipulations, we obtain:

$$\boldsymbol{e}^{(k+1)} = \boldsymbol{\theta}^{(k)}\boldsymbol{e}^{(k)} + \boldsymbol{\xi}^{(k)}\boldsymbol{C_2}\boldsymbol{e}^{(k)^2} + \boldsymbol{O}\left(\boldsymbol{e}^{(k)^3}\right) \tag{47}$$

where:

$$\boldsymbol{\theta}^{(k)} = \frac{1}{2}h^{(k)^2}\left(\psi^{(k)} - 2\right) + \frac{1}{2}h^{(k)}\left(\frac{1}{2}\psi^{(k)} - 3\right) + 1 \tag{48}$$

$$\boldsymbol{\xi}^{(k)} = 2h^{(k)^2}\left(\psi^{(k)} - 1\right) + \frac{3}{2}h^{(k)} - \frac{5}{4}\psi^{(k)} \tag{49}$$

For $\psi = 2$ and $h = 1$, (47) takes the form.

$$\boldsymbol{e}^{(k+1)} = \boldsymbol{C_2}\boldsymbol{e}^{(k)^2} + \boldsymbol{O}\left(\boldsymbol{e}^{(k)^3}\right) \tag{50}$$

Therefore, it is concluded that the developed HKW converges to $\boldsymbol{r}$ with quadratic convergence when $\psi = 2$ and $h = 1$. In the same way, it can be checked that the convergence of developed HKW is linear otherwise. Consequently, the proof is completed. □

Appendix B: Pseudocode of the developed HKW PF approach

Algorithm. Developed PF solution procedure based on the HKW

Let $\varepsilon > 0$, $\mu > 0$, $0 \le \psi^{(0)} \le 2$, $0 \le \bar{\psi} \le 2$, $0 < h_{\min} < h_{\max} \le 1$, $\alpha > 0$
    And $\boldsymbol{x}^{(0)}$ be given, $k \leftarrow 0$
while $\left\|\mathbf{g}\left(\boldsymbol{x}^{(k)}\right)\right\|_\infty > \varepsilon$ **do**
  if $k = 0$
    $SSR^{(0)} = \frac{1}{2}\left[\mathbf{g}\left(\boldsymbol{x}^{(0)}\right)\right]^T\mathbf{g}\left(\boldsymbol{x}^{(0)}\right)$
    $h^{(0)} \leftarrow \max\left\{h_{\min}, \min\left\{h_{\max}, {}^{1}/_{(SSR^{(0)})^{\mu}}\right\}\right\}$
  end if
  $\Delta\boldsymbol{x}^{(k)} = -\left[\mathbf{g}_x^{(k)}\right]^{-\mathbf{1}}\mathbf{g}\left(\boldsymbol{x}^{(k)}\right)$
  if $\psi^{(k)} < \bar{\psi}$
    $\boldsymbol{y}^{(k)} = \boldsymbol{x}^{(k)} + h^{(k)}\Delta\boldsymbol{x}^{(k)}$
    $\widetilde{\boldsymbol{x}}^{(k)} = \frac{1}{2}\left(\boldsymbol{x}^{(k)} + \boldsymbol{y}^{(k)}\right)$
    $\Delta\widetilde{\boldsymbol{x}}^{(k)} = -\left[\boldsymbol{g}_{\widetilde{x}}^{(k)}\right]^{-\mathbf{1}}\boldsymbol{g}\left(\widetilde{\boldsymbol{x}}^{(k)}\right)$
    $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \frac{h^{(k)}}{2}\left(\psi^{(k)}\Delta\boldsymbol{x}^{(k)} + \left(2 - \psi^{(k)}\right)\Delta\widetilde{\boldsymbol{x}}^{(k)}\right)$
  else if
    $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \Delta\boldsymbol{x}^{(k)}$
  end if
  if $\left\|\mathbf{g}\left(\boldsymbol{x}^{(k)}\right)\right\|_\infty \le \varepsilon$

$k \leftarrow k+1$

$SSR^{(k)} = \frac{1}{2}\left[\mathbf{g}\left(\boldsymbol{x}^{(k)}\right)\right]^T \mathbf{g}\left(\boldsymbol{x}^{(k)}\right)$

$\psi^{(k)} = 2\left|\frac{SSR^{(k)}-SSR^{(0)}}{SSR^{(0)}}\right|$

$\epsilon = \left\|\boldsymbol{x}^{(k)} - \boldsymbol{y}^{(k-1)}\right\|_\infty$

if $\epsilon > \alpha$

$h^{(k)} \leftarrow \max\{0{,}9h, h_{\min}\}$

else if

$h^{(k)} \leftarrow \min\{1{,}1h, h_{\max}\}$

end if

else

break

end if

end do

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