APPROXIMATION RATE AND SATURATION UNDER GENERALIZED CONVERGENCE

Dedicated to Professor Vijay Gupta on the occasion of his 60th birthday.

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Abstract. In this paper we prove a quantitative result about the convergence of sequences of functions defined from linear operators. The notion of convergence used here is the one given in [9]. The operators will be assumed to satisfy a shape preserving property associated with certain generalized derivative. We also study the saturation class, from the asymptotic condition that the sequence of operators fulfills. Finally, as applications, we show how the notion of weighted-g-statistical convergence, recently studied by A. Adem and M. Altinok [3], can be moved to the setting of approximation theory. Besides, we give a non standard example that shows the applicability of the results.

1. **Introduction.** In [9], the first author of the present paper establishes general conditions under which different notions of generalized convergence can be moved to the setting of approximation theory by linear operators. Thus, notions as A-summability or A-statistical convergence remain as particular cases of a general one. On the other hand, the approximation object there considered was certain generalized derivative of a function. He proved a qualitative Korovkin result, and showed conditions for the existence of an asymptotic condition. In addition to that, the study of the topic saturation started with the characterization of the so called trivial class. The present paper intends to be a natural continuation, thus completing the started path.

This paper is organized as follows. In section 2, we will show some required notions, and the notation will be set. In section 3, a quantitative Korovkin type result will be shown, as an extension of the qualitative one stated in [9]. In section 4, we will characterize the saturation class, thus finishing the study of this topic. In section 5 we will proceed to illustrate the way to translate the weighted statistical convergence, introduced by M. Balcerzak et al. [4], and recently studied by A. Adem and M. Altinok [3], to the setting of approximation theory. Finally, as a novelty

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with respect to usual examples found in the literature, we will define a sequence of linear operators following the seminal idea of J. P. King [11].

2. General Setting.

In this section, we will establish the framework, and present the required tools. Some notation will be set as well.

As usual, given $m \in \mathbb{N}$, $C^m[0,1]$ stands for the set of all functions m-times differentiable with continuous m-th derivative on the interval $[0,1]$. Notice that $C[0,1]$ is simply the set of all continuous functions defined on $[0,1]$ and $C^\infty[0,1] = \cap_{n\in\mathbb{N}} C^n[0,1]$.

Fixed a function $\tau \in C^\infty[0,1]$ such that $\tau(0)=0$, $\tau(1)=1$ and $\tau' > 0$ on $(0,1)$, consider the following differential operator for any $k \in \{1,2,...,m\}$ and $x \in [0,1]$:

$$D^k_\tau f(x) = D^k(f \circ \tau^{-1})(\tau(x)),$$

that is, the k-th derivative of the composition $f \circ \tau^{-1}$ evaluated at $\tau(x)$. Obviously, D^0_τ is the identity operator. In relation with the function τ, we define the functions $e^x_{\tau,l}$ given by

$$e^x_{\tau,l}(t) = (\tau(t) - \tau(x))^l \tag{1}$$

for any $t \in [0,1]$, $x \in [0,1]$ and each $l \in \mathbb{N}$. In the particular case $x = 0$, it is often written $e_{\tau,l}$ instead of $e^0_{\tau,l}$.

A straightforward computation provides the next useful identity

$$D^i_\tau e^x_{\tau,j} = \begin{cases} \dfrac{j!}{(j-i)!} e^x_{\tau,j-i} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Recall the definition of the classical modulus of continuity associated with a continuous function f,

$$\omega(f,\delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|, \quad \delta > 0,$$

and the following well-known property:

$$\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta), \quad \lambda,\delta > 0. \tag{3}$$

On the other hand, let $\mathcal{S}$ be the usual vector space of all real sequences, $\mathcal{S}_0$ some vector subspace of $\mathcal{S}$ and let $\mathfrak{L} : \mathcal{S}_0 \to \mathbb{R}$ be a linear functional fulfilling the following properties:

(A) If $x_n \in \mathcal{S}_0$ is convergent in the classical sense, then $\mathfrak{L}(x_n) = \lim x_n$.
(B) If $x_n, y_n \in \mathcal{S}_0$ and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $\mathfrak{L}(x_n) \leq \mathfrak{L}(y_n)$.
(C) If $a_n \in \mathcal{S}_0$ is non-negative with $\mathfrak{L}(a_n) = 0$, then $\mathfrak{L}(a_n x_n) = 0$ for any $x_n \in \mathcal{S}_0$.
(D) If $\{a_n\}, \{b_n\} \in \mathcal{S}_0$ with $\mathfrak{L}(a_n) = \mathfrak{L}(b_n) = \ell$ for every $n \in \mathbb{N}$ and $\{x_n\} \in \mathcal{S}$ is a sequence such that $a_n \leq x_n \leq b_n$ for every $n \in \mathbb{N}$, then $x_n \in \mathcal{S}_0$ and $\mathfrak{L}(x_n) = \ell$.
(E) If f is absolutely continuous and $\mathfrak{L}(x_n) = 0$ then $\mathfrak{L}(\omega(f,x_n)) = 0$

We show some definitions about rate of convergence related with the linear functional $\mathfrak{L}$. The first appears in [9] and the second is new one.

Definition 2.1. Given two sequences $x_n, y_n \in \mathcal{S}_0$,

i) $y_n = o_{\mathfrak{L}}(x_n)$ means that $\mathfrak{L}(x_n) = \mathfrak{L}(y_n) = \mathfrak{L}(x_n/y_n) = 0$.
ii) $y_n = O_{\mathfrak{L}}(x_n)$ means that $|x_n/y_n| \leq M + o_{\mathfrak{L}}(1)$ for some constant $M \geq 0$.

Remark 2.2. This convergence process generalizes some ones utilized in approximation theory until the date; for instance, the pointwise, weak, almost or statistical convergence. All of them lie perfectly in the general setting considered here. Concretely, when a new convergence process emerges from a theoretical point of view, one is always tempting to analyze it within approximation theory. The authors have observed that when this happens, people check many properties on the operators in order to get Korovkin-type results, among others. We have reduced this to only check a few properties, avoiding a repetitive effort to achieve the desired results.

3. **Quantitative Result in Generalized Convergence.** In the spirit of completeness that presides over this paper, a quantitative version of a Korovkin-type result is shown. This result allows us to establish the generalized convergence of the process from the generalized convergence of the test functions $\{e_{\tau,m}, e_{\tau,m+1}, e_{\tau,m+2}\}$. It also provides us with additional information on the rate of the approximation process. The result resembles closely Theorem 1 in [6], with the proper changes.

Theorem 3.1. *Assume that $m \in \mathbb{N}$ and that $\{L_n\}$ is a sequence of linear operators from $C^m[0,1]$ into itself fulfilling the next additional shape-preserving condition:*

$$D_\tau^m f \geq 0 \Rightarrow D_\tau^m L_n f \geq 0. \tag{4}$$

If $f \in C^m[0,1]$ and $x \in [0,1]$, then this inequality holds true for every $n \in \mathbb{N}$:

$$\begin{aligned} |D_\tau^m L_n f(x) - D_\tau^m f(x)| \leq \frac{|D_\tau^m f(x)|}{m!}\ |D_\tau^m L_n e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)| \\ + \frac{1}{m!}\ |D_\tau^m L_n e_{\tau,m}(x) + D_\tau^m e_{\tau,m}(x)|\ \omega(D_\tau^m f \circ \tau^{-1}, \eta_{n,m,\tau}(x)), \end{aligned} \tag{5}$$

where

$$\eta_{n,m,\tau}^2(x) := \frac{2}{(m+2)!}|D_\tau^m L_n e_{\tau,m+2}^x(x)|.$$

Proof. Let $f \in C^m[0,1]$, $x \in [0,1]$ and let $\xi > 0$ be an arbitrary fixed number to be determined later. On the one hand, suppose that $t \in [0,1]$ satisfies $|\tau(t) - \tau(x)| \leq \xi$, then the definition of the modulus of continuity allows to write the following:

$$\begin{aligned} &|(D_\tau^m f \circ \tau^{-1})(\tau(t)) - (D_\tau^m f \circ \tau^{-1})(\tau(x))| \leq \\ &\leq \sup_{|r-s| \leq |\tau(t)-\tau(x)|} |(D_\tau^m f \circ \tau^{-1})(r) - (D_\tau^m f \circ \tau^{-1})(s)| \\ &\leq \sup_{|r-s| \leq \xi} |(D_\tau^m f \circ \tau^{-1})(r) - (D_\tau^m f \circ \tau^{-1})(s)| \\ &= \omega(D_\tau^m f \circ \tau^{-1}, \xi) \leq \left(1 + \frac{|\tau(t) - \tau(x)|^2}{\xi^2}\right) \omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right). \end{aligned}$$

On the other hand, suppose that $t \in [0,1]$ satisfies $|\tau(t) - \tau(x)| > \xi$, then the definition of the modulus of continuity together with the mentioned property (3) allow to write

$$\begin{aligned}
&|(D_\tau^m f\circ\tau^{-1})(\tau(t)) - (D_\tau^m f\circ\tau^{-1})(\tau(x))|\\
&\le \sup_{|r-s|\le|\tau(t)-\tau(x)|} |(D_\tau^m f\circ\tau^{-1})(r) - (D_\tau^m f\circ\tau^{-1})(s)|\\
&= \omega(D_\tau^m f\circ\tau^{-1}, |\tau(t)-\tau(x)|) = \omega\left(D_\tau^m f\circ\tau^{-1}, \frac{|\tau(t)-\tau(x)|}{\xi}\ \xi\right)\\
&= \left(1+\frac{|\tau(t)-\tau(x)|}{\xi}\right)\omega\left(D_\tau^m f\circ\tau^{-1},\xi\right)\\
&\le \left(1+\frac{|\tau(t)-\tau(x)|^2}{\xi^2}\right)\omega\left(D_\tau^m f\circ\tau^{-1},\xi\right).
\end{aligned}$$

In other words,

$$|(D_\tau^m f\circ\tau^{-1})(\tau(t)) - (D_\tau^m f\circ\tau^{-1})(\tau(x))| \le \left(1+\frac{|\tau(t)-\tau(x)|^2}{\xi^2}\right)\omega\left(D_\tau^m f\circ\tau^{-1},\xi\right).$$

Rewriting $(D_\tau^m f\circ\tau^{-1})(\tau(t)) - (D_\tau^m f\circ\tau^{-1})(\tau(x))$ as $D_\tau^m\left[f - D_\tau^m f(x)\ \frac{e_{\tau,m}}{m!}\right]$ and taking into account (1) and (2), give the following expression

$$\begin{aligned}
-\omega\left(D_\tau^m f\circ\tau^{-1},\xi\right)\left[1+\frac{e_{\tau,2}^x(t)}{\xi^2}\right] &\le D_\tau^m\left[f - D_\tau^m f(x)\ \frac{e_{\tau,m}}{m!}\right]\\
&\le \omega\left(D_\tau^m f\circ\tau^{-1},\xi\right)\left[1+\frac{e_{\tau,2}^x(t)}{\xi^2}\right].
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&-\omega\left(D_\tau^m f\circ\tau^{-1},\xi\right) D_\tau^m\left[\frac{e_{\tau,m}}{m!}+\frac{2e_{\tau,m+2}^x}{(m+2)!\xi^2}\right]\\
&\le D_\tau^m\left[f - D_\tau^m f(x)\ \frac{e_{\tau,m}}{m!}\right]\\
&\le \omega\left(D_\tau^m f\circ\tau^{-1},\xi\right) D_\tau^m\left[\frac{e_{\tau,m}}{m!}+\frac{2e_{\tau,m+2}^x}{(m+2)!\xi^2}\right].
\end{aligned}$$

The next step is to apply the operator L_n to the previous inequalities, using the shape-preserving condition:

$$\begin{aligned}
-\omega\left(D_\tau^m f\circ\tau^{-1},\xi\right) D_\tau^m L_n\left[\frac{e_{\tau,m}}{m!}+\frac{2e_{\tau,m+2}^x}{(m+2)!\xi^2}\right] &\le D_\tau^m\left[L_n f - D_\tau^m f(x)\ \frac{L_n e_{\tau,m}}{m!}\right]\\
&\le \omega\left(D_\tau^m f\circ\tau^{-1},\xi\right) D_\tau^m L_n\left[\frac{e_{\tau,m}}{m!}+\frac{2e_{\tau,m+2}^x}{(m+2)!\xi^2}\right]. \qquad (6)
\end{aligned}$$

Finally, thanks to the triangular inequality

$$|D_\tau^m L_n f(x) - D_\tau^m f(x)| =$$

$$= \left| \left(D_\tau^m L_n f(x) - D_\tau^m f(x) \tfrac{D_\tau^m L_n e_{\tau,m}(x)}{m!} \right) + \left(D_\tau^m f(x) \tfrac{D_\tau^m L_n e_{\tau,m}(x)}{m!} - D_t^m f(x) \right) \right|$$

$$\leq \underbrace{\left| D_\tau^m L_n f(x) - D_\tau^m f(x) \tfrac{D_\tau^m L_n e_{\tau,m}(x)}{m!} \right|}_{A} + \underbrace{\left| D_\tau^m f(x) \tfrac{D_\tau^m L_n e_{\tau,m}(x)}{m!} - D_\tau^m f(x) \right|}_{B},$$

but making use of (6) in A and extracting common factor in B, we get

$$A \leq \omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right) D_\tau^m L_n \left[\frac{e_{\tau,m}}{m!} + \frac{2e^x_{\tau,m+2}}{(m+2)!\xi^2} \right],$$

$$B = \frac{|D_\tau^m f(x)|}{m!} \, |D_\tau^m L_n e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)|.$$

As a consequence, taking $\xi = \eta_{n,m,\tau}(x)$ and again writing $1 = \frac{D_\tau^m e_{\tau,m}(x)}{m!}$, follows the desired result:

$$|D_\tau^m L_n f(x) - D_\tau^m f(x)|$$

$$\leq \tfrac{|D_\tau^m f(x)|}{m!} \, |D_\tau^m L_n e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)|$$

$$+\omega\left(D_\tau^m f \circ \tau^{-1}, \xi\right) D_\tau^m L_n \left[\tfrac{e_{\tau,m}}{m!} + \tfrac{2e^x_{\tau,m+2}}{(m+2)!\xi^2} \right]$$

$$= \tfrac{|D_\tau^m f(x)|}{m!} \, |D_\tau^m L_n e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)|$$

$$+\omega\left(D_\tau^m f \circ \tau^{-1}, \eta_{n,m,\tau}(x)\right) \left[D_\tau^m L_n \tfrac{e_{\tau,m}}{m!} + 1 \right]$$

$$= \tfrac{|D_\tau^m f(x)|}{m!} \, |D_\tau^m L_n e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)|$$

$$+\omega\left(D_\tau^m f \circ \tau^{-1}, \eta_{n,m,\tau}(x)\right) \left[\tfrac{D_\tau^m L_n e_{\tau,m}}{m!} + \tfrac{D_\tau^m e_{\tau,m}(x)}{m!} \right].$$

□

The qualitative result in the general setting mentioned in section 1, which is an extension of the classical result of Korovkin, was established in [9]. Here, we have just presented a quantitative result, from which the qualitative one is easily derived.

We will clarify what was said in the introduction to this section. From Theorem 3.1 and taking into account that our linear functional satisfies both properties (D) and (E) it is easy to deduce the convergence of the process from the convergence on the test functions. It is revealed in the following corollary.

Corollary 3.2. *If* $\mathfrak{L}(D_\tau^m L_n e_{\tau,m+k}(x) - D_\tau^m e_{\tau,m+k}(x)) = 0$ *for* $k \in \{1,2,3\}$*, then* $\mathfrak{L}(D_\tau^m L_n f(x) - D_\tau^m f(x)) = 0$ *for any function* $f \in C^m[0,1]$.

The following result, about the rate of the process, is derived from Definition 2.1.

Corollary 3.3. *Suppose that* $f \in C^m[0,1]$*,* $x \in [0,1]$ *and* $\{a_n\}, \{b_n\} \in \mathcal{S}_0$ *are two sequences with* $\mathfrak{L}(a_n) = \mathfrak{L}(b_n) = 0$*. Assume that* $c_n := \max\{a_n, b_n\}$ *for each* $n \in \mathbb{N}$*. If* $\{D_\tau^m L_n e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)\} = o_{\mathfrak{L}}(a_n)$ *and* $\{\omega(D_\tau^m f \circ \tau^{-1}, \eta_{n,m,\tau}(x))\} = o_{\mathfrak{L}}(b_n)$*, then* $\{D_\tau^m L_n f(x) - D_\tau^m f(x)\} = o_{\mathfrak{L}}(c_n)$.

Proof.

$$\begin{aligned}\frac{|D_\tau^m L_n f(x) - D_\tau^m f(x)|}{c_n} &\le \frac{|D_\tau^m f(x)|}{m!}\frac{|D_\tau^m L_n e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)|}{c_n}\\ &+\frac{1}{m!}\,|D_\tau^m L_n e_{\tau,m}(x) + D_\tau^m e_{\tau,m}(x)|\frac{\omega(D_\tau^m f\circ\tau^{-1},\eta_{n,m,\tau}(x))}{c_n}\\ &\le \frac{|D_\tau^m f(x)|}{m!}\frac{|D_\tau^m L_n e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)|}{a_n}\\ &+\frac{1}{m!}\,|D_\tau^m L_n e_{\tau,m}(x) + D_\tau^m e_{\tau,m}(x)|\frac{\omega(D_\tau^m f\circ\tau^{-1},\eta_{n,m,\tau}(x))}{b_n}.\end{aligned}$$

Then, using the appropriate properties, the desired result is reached. □

4. **Saturation class.** From now onwards, we will assume that there exist a sequence of positive real numbers $\{\lambda_n\}$ with $\mathfrak{L}(\lambda_n) = 0$ and three strictly positive functions $w_k \in C^{2-k}(0,1)$ for each $k \in \{0,1,2\}$ such that for any $f \in C^m[0,1]$ ($m+2$ times differentiable in some neighborhood of $x \in (0,1)$),

$$\mathfrak{L}\left(\frac{D_\tau^m L_n f(x) - D_\tau^m f(x)}{\lambda_n}\right) = w_2^{-1}D^1(w_1^{-1}D^1(w_0^{-1}D_\tau^m\ f)(x). \quad (7)$$

From the existence of an asymptotic condition we can see that the rate of convergence is limited, despite the smoothness of the function. In [9] a characterization of the functions such that $D_\tau^m L_n f(x) - D_\tau^m f(x) = o_{\mathfrak{L}}(\lambda_n)$ is given. This set of functions is well known as trivial class. We complete this study obtaining an analogous result with the saturation class based on the definitions shown above.

Theorem 4.1. *Assuming $f \in C^m[0,1]$, the next two statements are equivalent:*

(a) $\dfrac{|D_\tau^m L_n f(x) - D_\tau^m f(x)|}{\lambda_n} \le M + o_{\mathfrak{L}}(1)$ *for all* $x \in (0,1)$.

(b) $w_2^{-1}D^1\left(w_1^{-1}D^1\left(w_0^{-1}D_\tau^m f\right)\right) \le M$ *a.e. on* $(0,1)$.

We show the following results that are required for the proof of the theorem.

Lemma 4.2 (see [5, Lemma 1])**.** *Let J be an open subinterval of $[0,1]$, and let a, b, x_0 be three points of J with $a < x_0 < b$. If $g, h \in C[0,1]$ are two continuous function on J such that $g(a) = g(b) = 0$ and $g(x_0) > 0$, then there exists a real number $\alpha < 0$, a solution z of the ordinary differential equation on J*

$$w_2^{-1}D^1(w_1^{-1}D^1(w_0^{-1}D_\tau^m\ z) = 0, \text{ where } w_k \in C^{2-k}(0,1) \text{ for each } k \in \{0,1,2\}, \quad (8)$$

and a point $\xi \in (a,b)$ such that

$$\alpha h(\xi) + z(\xi) = g(\xi) \text{ and } \alpha f(t) + z(t) \ge f(t) \text{ for all } t \in (a,b).$$

Lemma 4.3 (see [9, Lemma 5.1])**.** *Assume $h \in C^m[0,1]$ and $x \in (0,1)$. If there exists a neighborhood N_x of x where $D_\tau^m h \ge 0$, then $D_\tau^m L_n h(x) \ge 0 + o_{\mathfrak{L}}(\lambda_n)$.*

Lemma 4.4 (see [9, Lemma 5.3])**.** *Assume $a, b \in (0,1)$ with $a < b$ and $x \in (a,b)$. A function $f \in C^m[0,1]$ is a solution of the ordinary differential equation (8) in some neighborhood of x if, and only if, $D_\tau^m L_n f(x) - D_\tau^m f(x) = o_{\mathfrak{L}}(\lambda_n)$.*

As we can see in [9], the ordinary differential equation associated with the asymptotic condition

$$w_2^{-1}D^1(w_1^{-1}D^1(w_0^{-1}D_\tau^m\ y)) \equiv 0$$

can be reduced to another one of second order, namely $\omega_2^{-1}D^1\left(\omega_1^{-1}D^1\left(\omega_0^{-1}z\right)\right) \equiv 0$. Now, let $\{u,v\}$ be a fundamental set of solutions that forms also an extended complete Tchebychev system. The following two propositions, where the notion of convexity with respect to $\{u,v\}$ plays an important role, prepare the way to prove the desired result.

Proposition 4.5. *Assuming* $f \in C^m[0,1]$, $D_\tau^m f$ *is convex with respect to* $\{u,v\}$ *on* $(0,1)$ *if, and only if,* $D_\tau^m L_n f(x) \geq D_\tau^m f(x) + o_{\mathfrak{L}}(\lambda_n)$ *for any* $x \in (0,1)$.

Proof. Suppose that $x \in (0,1)$, $D_\tau^m f$ is convex with respect to $\{u,v\}$ on $(0,1)$ and take $z \in \langle u,v\rangle$ such that $z(x) = D_\tau^m f(x)$ and $D^1 z(x) = D_+^1(D_\tau^m f)(x)$, where D_+^1 denotes the right first derivative operator. Lemma 2.2 in [12] provides that $z(t) \leq D_\tau^m f(t)$ for all $t \in (0,1)$, but taking $Z \in C^m[0,1]$ such that $D_\tau^m Z = z$ on $(0,1)$ and applying Lemma 4.3, we get

$$D_\tau^m L_n Z(x) \leq D_\tau^m L_n f(x) + o_{\mathfrak{L}}(\lambda_n)$$

or, equivalently, since $D_\tau^m Z(x) = z(x) = D_\tau^m f(x)$,

$$D_\tau^m L_n Z(x) - D_\tau^m Z(x) \leq D_\tau^m L_n f(x) - D_\tau^m f(x) + o_{\mathfrak{L}}(\lambda_n).$$

Finally, Lemma 4.4, applied to the function Z, gives the desired inequality.
In order to prove the converse, we assume the contrary, i.e. $D_\tau^m f$ is not convex with respect to $\{u,v\}$ on $(0,1)$. In that case, there exist three points $a,b,\xi \in (0,1)$ with $a < \xi < b$ and $F(D_\tau^m f,a,b)(\xi) < D_\tau^m f(\xi)$, where $F(D_\tau^m,a,b)$ stands for the unique function of the space $\langle u,v\rangle$ which interpolates D_τ^m at a and b. Lemma 4.2, with $\phi = D_\tau^m f - F(D_\tau^m f,a,b)$ and $\psi = v$, provides/assures the existence of $\varepsilon < 0$, a solution $\widehat{z}$ of (8) and a point $x_0 \in [a,b]$ such that for each $t \in (a,b)$,

$$\varepsilon v(t) + \widehat{z}(t) \geq D_\tau^m f(t) - F(D_\tau^m f,a,b)(t) \tag{9}$$

and

$$\varepsilon v(x_0) + \widehat{z}(x_0) = D_\tau^m f(x_0) - F(D_\tau^m f,a,b)(x_0). \tag{10}$$

Given $V, \widehat{Z}, \widehat{F} \in C^m[a,b]$ such that $D_\tau^m V = v$, $D_\tau^m \widehat{Z} = \widehat{z}$ and $D_\tau^m \widehat{F} = F(D_\tau^m f,a,b)$ on (a,b), we make use of Lemma 4.3 taking into account (9):

$$\varepsilon D_\tau^m L_n V(x_0) + D_\tau^m L_n \widehat{Z}(x_0) \geq D_\tau^m f(x_0) - D_\tau^m \widehat{F}(x_0) + o_{\mathfrak{L}}(\lambda_n).$$

In addition, plugging the equality (10) into the inequality from above yields

$$\varepsilon\left[D_\tau^m L_n V(x_0) - D_\tau^m V(x_0)\right] + D_\tau^m L_n \widehat{Z}(x_0) - D_\tau^m \widehat{Z}(x_0) \geq$$

$$\geq D_\tau^m L_n f(x_0) - D_\tau^m f(x_0) - \left[D_\tau^m L_n \widehat{F}(x_0) - D_\tau^m \widehat{F}(x_0)\right] + o_{\mathfrak{L}}(\lambda_n).$$

Finally, dividing by λ_n and using the hypothesis (7), we obtain the following inequality which contradicts our assumptions:

$$\varepsilon \geq D_\tau^m L_n f(x_0) - D_\tau^m f(x_0) + o_{\mathfrak{L}}(\lambda_n).$$

□

Proposition 4.6. *If* $M \geq 0$ *and* $f,g \in C^m[0,1]$, *then these two statements are equivalent:*

(a) $M D_\tau^m g \pm D_\tau^m f$ *are convex with respect to* $\{u,v\}$ *on* $(0,1)$.
(b) $|D_\tau^m L_n f(x) - D_\tau^m f(x)| \leq M\left(D_\tau^m L_n g(x) - D_\tau^m g(x)\right) + o_{\mathfrak{L}}(\lambda_n),\ 0 < x < 1$.

Proof. It is easily derived from Proposition 4.5 replacing f by $Mg \pm f$.

□

In the last term, for proving the main result of this section, we apply Proposition 4.6 with the following choice $g(x) \in C^m[a,b]$ such that

$$D_\tau^m g(x) = \omega_0(x) \int_c^x \omega_1(x_1) \int_c^{x_1} \omega_2(x_2) dx_2 dx_1.$$

From the asymptotic condition we have $\mathfrak{L}\left(\frac{D_\tau^m L_n g(x) - D_\tau^m g(x)}{\lambda_n}\right) = 1$ and finally we use the characterization of Theorem 3.2 in [12].

5. **Applications.**

When a new notion of generalized convergence arises and later is transferred to an approximation theory setting, it usually receives a particular and specific treatment. We can see some illustrative instances in [8] and [10]. In this section, we show a different approach to proceed with such a sort of transfers. As an application of the previous results, we define a sequence of operators by, say, a non standard method, as it differs from those based on considering sequences of the type $(1+\alpha_n)L_n$, α_n being a sequence of real numbers satisfying the specific convergence property.

On the other hand, the results of Section 4 can also be applied to further sequences of linear operators that appears after considering usual modifications of Kantorovich and Durrmeyer type, see the instance the recent papers [13], [2].

5.1. **Weighted-statistical convergence.**

In 2020, Abdu Awel Adem and Maya Altinok [3] provided us with a new notion of generalized convergence, the weighted-g-statistical convergence. The purpose of this section is to show that this new notion verifies the conditions listed above, and in this way it can be transferred to the theory of approximation theory. We recall some definitions.

Let $g : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} n/g(n) \neq 0$. This function is called a weight function.

Definition 5.1. Let $A \subset \mathbb{N}$ and $A(n) = \{k \in \mathbb{N} : k \leq n\}$. The weight g density of A is defined by

$$d_g(A) = \lim_{n\to\infty} \frac{|A(n)|}{g(n)}.$$

If x_k is a sequence that satisfies property P for all k except a set of weight g density zero, then we say that x_k satisfies P for (weight g almost all k) and it is denoted by $(g - a.a.k)$ for simplicity. As a consequence we have that if $d_g(A) = d_g(B) = 0$ then $d_g(A \cup B) = 0$.

Based on the above, we proceed to define the notion of generalized convergence.

Definition 5.2. A real sequence x_k is weighted g-statistical convergent to a number L if for each $\epsilon > 0$,

$$d_g(K_\epsilon) = 0, \text{ where } K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}.$$

In this case we write $g - st - \lim x_k = L$

C_g^{st} denotes the set of all weighted g-statistical convergent sequences.

The following characterization will be very useful.

Lemma 5.3 (see [3])**.** *The following statements are equivalent:*

(i) $g - st - \lim x_k = L$

(ii) x_k *is a sequence for which there exists a sequence* y_k *convergent to* L *such that* $x_k = y_k,\ (g - a.a.k)$.

We consider the following functional on the set of all weighted g-statistical convergent sequences:

$$\mathfrak{L} : C_g^{st} \to \mathbb{R}, \quad \mathfrak{L}(x_k) = L,\ g - st - \lim x_k = L$$

As we can see, this is a linear functional. We suppose that $x_k, y_k \in C_g^{st}$ and $g - st - \lim x_k = L_1$, $g - st - \lim y_k = L_2$. Then there exist both sequences $\tilde{x}_k$ and $\tilde{y}_k$ which converge towards L_1 and L_2 respectively, such that $x_k = \tilde{x}_x,\ (g - a.a.k)$ and $y_k = \tilde{y}_k,\ (g - a.a.k)$. In this case $\tilde{x}_k + \tilde{y}_k$ is convergent towards $L_1 + L_2$ and $x_k + y_k = \tilde{x}_k + \tilde{y}_k,\ (g - a.a.k)$ which proves that:

$$\mathfrak{L}(x_k + y_k) = \mathfrak{L}(x_k) + \mathfrak{L}(y_k).$$

Analogously, $\mathfrak{L}(\lambda x_k) = \lambda \mathfrak{L} x_k,$ with $\lambda \in \mathbb{R}$.

Similarly, we can prove conditions (A), (B), (C), (D), (E). As a conclusion the weight g-statistical convergence verifies all the conditions to be moved to the setting of approximation theory.

5.2. **Example.** Following the ideas in [11], [7] and [1], we define the following sequence of operators derived from the classical Bernstein operators and the function τ given in a previous section:

$$\widetilde{B}_n^{\tau} f(x) = \sum_{k=0}^{n} \binom{n}{k} \tau(r_n(x))^k (1-\tau(r_n(x)))^{n-k} (f \circ \tau^{-1})(k/n), \quad f \in C[0,1],\ x \in [0,1], \tag{11}$$

where the sequence $r_n(x)$ is defined by

$$r_n(x) = x s_n, \qquad s_n = \begin{cases} 1 & n \neq m^2 \\ 0 & n = m^2. \end{cases}$$

More explicitly,

$$\{r_n(x)\} = \{0, x, x, 0, x, x, x, x, 0, x, x, x, x, x, x, 0, x, x, x, x, x, x, x, x, 0, ...\}.$$

It is easy to see that $g - st - \lim r_n(x) = x$ with weight function $g(n) = \log(n)$.

We have the following $\widetilde{B}_n^{\tau} f(x) = B_n^{\tau} f(x)\ (g - a.a.k)$. In [7] we can see that B_n^{τ} verifies the shape-preserving condition displayed in the expression (4).

As a conclusion, the weighted-g-statistical notion of convergence can be moved to approximation theory, and the sequence of operators fulfills an adequate shape preserving condition, so that we can apply all the results of the preceding sections.

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