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Comprehensive minimum cost models for large scale group decision making with consistent fuzzy preference relations

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ABSTRACT

Nowadays, society demands group decision making (GDM) problems that require the participation of a large number of experts, so-called large scale group decision making (LS-GDM) problems. Logically, the more experts are involved in the decision making process, the more common is the emergence of disagreements in the group. For this reason, consensus reaching processes (CRPs) are key in the resolution of these problems in order to smooth such disagreements in the group and reach consensual solutions. A CRP requires that experts are receptive to change their initial preferences, but demanding excessive changes could lead to deadlocks. The well-known minimum cost consensus (MCC) model allows to obtain an agreed solution by preserving experts' preferences as much as possible. However, this MCC model only considers the distance among experts and collective opinion, which is not enough to guarantee a desired degree of consensus. To overcome this limitation, it was proposed comprehensive MCC models (CMCC) in which both consensus degree and distance are considered, and CMCC models deal with fuzzy preference relations (FPRs) for modeling experts' opinions. However, these models are not efficient to deal with LS-GDM problems and the FPRs consistency is ignored in them. Therefore, this paper aims to propose new CMCC models focused on LS-GDM problems in which experts use FPRs whose consistency is taken into account in order to obtain reliable results. A case study is introduced to show the effectiveness of the proposed models.

1. Introduction

Human beings spend much of their time making decisions, what to wear, what to eat today, which political party to vote for and so on. In general terms, a decision making (DM) problem is characterized by a set of alternatives and the need of choosing one or several of them as solution of the problem. However, either because of the increasing complexity of the decisions or the need to consider different points of view for making decisions, group decision making (GDM) has become a fundamental activity in any society. In a GDM problem, several experts with their own attitudes provide their preferences over the alternatives with the aim to achieve a common solution [1].

Society is constantly evolving, the emergence of new technological advances such as social networks services [2] or big data [3] and societal needs such as e-democracy [4] or e-marketplace selection [5] have given place to a new type of GDM problems so-called large scale group decision making (LS-GDM) problems [6–9]. Whereas classical GDM problems require the participation of just a few experts, the number of experts in LS-GDM problems is much larger. In spite of there is no a consensus in the specialized literature about when a GDM problem is LS-GDM [10], it has been considered LS-GDM when the number of experts engaged in the decision process is greater than 20 [10], although the number might be hundreds or thousands.

The classical solving process for GDM problems [11] does not consider the level of agreement among experts before the selection process of the best alternative. Consequently, some experts might not agree with the final solution and feel that their opinions have been ignored [12,13]. The latter issue is particularly relevant in LS-GDM problems, since they usually are related to decisions that affect directly society where consensual solutions are more appreciated. For this reason, consensus reaching processes (CRPs) [13] are included as an additional phase in the classical solving process for GDM problems. A CRP is a cyclical process in which experts discuss and modify their initial preferences throughout different discussion rounds in order to increase the level of agreement and obtain an acceptable solution for all of them. Despite the fact, the greater the number of experts in a GDM problem, the greater the number of conflicts and disagreements that may arise among them and consequently, a greater

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necessity of applying a CRP, most of CRPs proposals are focused on GDM problems with a few number of experts [14].

A CRP usually involves changes in the initial experts' opinions. However, some experts may be reluctant to modify their preferences and demanding excessive modifications in order to reach consensus might lead to deadlocks. Several researchers have pointed out the importance of considering cost of changing experts' opinion to achieve consensus and it has become an attractive challenge to tackle in CRPs [15–18]. Ben-Arieh and Easton [15] defined the concept of minimum-cost consensus (MCC) and based on this concept Zhang et al. [18] proposed a MCC model that makes use of a linear function whose aim is to obtain a consensual solution by minimizing the cost of changing experts' initial opinions and preserving them as much as possible. Afterwards, several researchers have proposed new MCC models [17–22]. Nevertheless, the way of computing consensus in the previous approaches, by considering only the distance among experts and collective opinion, does not always guarantee to achieve an acceptable level of agreement in the group [16]. For this reason, Labella et al. proposed in [16] new comprehensive MCC (CMCC) models that consider both distance among experts to collective opinion and a minimum agreement among all experts. In this way, it is possible to guarantee to reach a desired level of consensus. Nevertheless, the proposed MCC models are focused on GDM problems with a few experts, by ignoring LS-GDM problems in which CRPs are more important and necessary.

Therefore, it seems necessary to improve the integration of MCC based CRPs into LS-GDM to smooth the conflicts and opinions polarization [23] that usually appear in this type of problems in real-world. Hence, in this research we aim at developing new CMCC models able to cope with the scalability that appears in LS-GDM problems both from programming and time costs, and guarantee a consensual solution for the group. Last but not least, our models should be able to deal with different preference modeling structures that are used in LS-GDM. Initially, CMCC approaches have been developed to deal with numerical utility vectors, but the spread use of fuzzy preference relations [24] (FPRs) in GDM problems drive us to extend our new proposals for CMCC in LS-GDM to manage also this type of preference representation. Furthermore, consistency in FPRs is key to obtain reliable results, thus the new CMCC models for LS-GDM problems will consider the consistency in the experts' preferences.

To sum up and clarify our research, the main aims of this contribution are the following ones:

- To establish the relation between MCC and CMCC models.
- To establish the connection between the two widely used consensus measures and characterize the relation between the CMCC models based on these consensus measures.
- To define novel CMCC based CRP models for LS-GDM that guarantee consensual solution in spite of scalability problem.
- To extend previous novel models to deal with the consistency in FPRs.
- To analyze applicability of these CMCC models and dictate the setup that requires for solving CMCC models in LS-GDM scenario (involving more than 100 experts) with the available solver tools, and illustrate with a didactic example along with its comparative advantages.
- To advocate the implementation of the proposed CMCC models in real time application of LS-GDM, complexity and scale-ability of the different types of solver tools for solving the proposed models are analyzed.
- To advocate future research in LS-GDM, a few simulated data-sets with the CMCC solutions are provided, which will allow the researchers to compare the performance of proposed CRP process in terms of minimum cost consensus metric given in [16].

The remaining of this contribution is structured as follows: Section 2 reviews some basic concepts in order to facilitate the understanding of the proposal. Section 3 introduces novel CMCC approaches for LS-GDM. Section 4 shows the usefulness of the proposed CMCC based CRPs in a LS-GDM problem, includes a comparative performance analysis and studies their complexity. Finally, in Section 5 some conclusions are drawn.

2. Preliminaries

This section reviews some concepts related to CRPs, LS-GDM and MCC approaches to facilitate the understanding of the proposal.

2.1. CRPs and LS-GDM

In real world people have to make decisions since they get up and select which clothes they will wear that day. There are problems in the daily life in which it is very hard for a single person to assess all the possible alternatives to make a decision. This leads to GDM problems in which a handful of experts with different points of view and knowledge provide their preferences over a set of alternatives, to obtain a common solution [25]. Nevertheless, nowadays with the technological development and social demands [26,27], the number of experts involved in the decision problems is increasing and thus, a new concept has arisen in the DM field, LS-GDM. The main difference with the classical GDM problems is the number of experts, because LS-GDM problems deal with a large number of experts [10]. In this paper we may assume thousands of experts.

Formally, a LS-GDM problem is defined as follows: (i) a set of alternatives $X = \{x_1, \ldots, x_n\}$, $(n \geq 2)$, that can be selected as possible solutions for the problem, and (ii) a set of experts $E = \{e_1, \ldots, e_m\}$, $(m \gg n)$, who elicit their assessments on the set of alternatives X.

A LS-GDM problem can be solved in a similar way to the traditional GDM problems by a selection process which is divided into two phases: (i) aggregation: the preferences elicited by experts are fused by means of an aggregation operator to get a collective opinion, and (ii) exploitation: the best alternative is selected taking into account the collective opinion obtained in the previous phase. Notwithstanding, this process does not always guarantee to obtain a solution accepted by all experts and some of them might feel that their opinions were not considered. In LS-GDM problems these situations are much more common, because opinions among a large number of experts tend to be easily conflicting. To avoid these situations, a CRP is included before the selection process.

A CRP is a dynamic process in which experts discuss and modify their preferences to reach a consensus that is accepted by the whole group of experts [28]. Different consensus models can be found in the literature [14,29,30]. A general scheme of a CRP is shown in Fig. 1 and it is described as follows:

- *Computing consensus*: Once experts have provided their preferences, a consensus degree μ, is computed by using consensus measures [31].
- *Consensus control*: The consensus degree is compared with the consensus threshold set previously, $\alpha \in [0, 1]$. If the consensus degree reached is equal to or greater than the consensus threshold, $\mu \geq \alpha$, the selection process is applied, otherwise, it is necessary a discussion round to increase the consensus degree. In order to avoid an infinite number of rounds, a parameter that denotes the maximum number of rounds allowed is defined, $max\_round \in \mathbb{N}$.

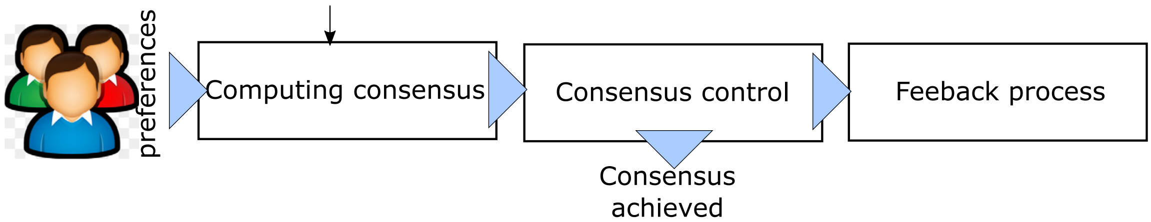

Fig. 1. General scheme of a CRP.

Table 1
Consensus achieved by using (M-2).

ε	μ
0.5	0.58
0.15	0.83

- *Feedback process*: The furthest experts of the collective opinion are identified to suggest them to modify their preferences and increase the consensus degree in the next round.

In the last few years, some scholars have paid attention on LS-GDM in CRPs and several proposals have been presented in the literature [6,7,32–34] to cope with same challenges such as, time cost, scalability, behavior, etc.

2.2. Minimum cost consensus

Several scholars have pointed out that in CRPs the cost of modifying experts' opinions in the collective opinion is very important. For this reason, some MCC models have been proposed [15, 18,35]. The concept of MCC was introduced by Ben-Arieh and Easton [15] as follows, *"the consensus is reached when the distance between experts and the collective opinion is minimum"*. Taking as basis this concept, Zhang et al. [18] proposed a MCC model as follows:

Definition 1 (*[18]*)**.** Let $(o_1, \ldots, o_m)$ be the assessments provided by a set of experts $E = \{e_1, \ldots, e_m\}$ over an alternative, and c_k be the cost of moving expert e_k's opinion 1 unit. The MCC model based on a linear cost function is given as follows:

$$(\mathbf{M-1})$$
$$\min \sum_{k=1}^{m} c_k|\bar{o}_k - o_k|$$
$$s.t.\ |\bar{o}_k - \bar{o}| \leq \varepsilon, k \in \mathbb{I}_m,$$

where $(\bar{o}_1, \ldots, \bar{o}_m)$ are the adjusted experts' opinions, $\bar{o}$ is the collective opinion, ε is the maximum acceptable distance of each expert to the collective opinion and $\mathbb{I}_m = \{1, 2, \ldots, m\}$ is the index set corresponding to the set of m experts $E = \{e_1, e_2, \ldots, e_m\}$. At the abstraction level m maps to the index set $\{1, 2, \ldots, m\}$, i.e., $\mathbb{I}_m$.

According to this model, an expert does not need to change his/her opinion, if it is in the interval $[\bar{o}-\varepsilon, \bar{o}+\varepsilon]$, and any opinion further than ε from $\bar{o}$ should only be changed until that expert's opinion is exactly ε away from $\bar{o}$.

Based on the MCC model (M-1), Zhang et al. [18] studied how the aggregation operator used to fuse experts' opinions and obtain the collective opinion can affect in the computation of the consensus level. They focused on weighted average (WA) operator and ordered weighted average (OWA) operator and proposed a MCC model described as follows:

$$(\mathbf{M-2})$$
$$\min \sum_{k=1}^{m} c_k|\bar{o}_k - o_k|$$
$$\text{s.t.} \begin{cases} \bar{o} = A(\bar{o}_1, \bar{o}_2, \ldots, \bar{o}_m) \\ |\bar{o}_k - \bar{o}| \leq \varepsilon,\ k \in \mathbb{I}_m \end{cases}$$

where $A : [0, 1]^n \rightarrow [0, 1]$ *is an aggregation operator used to obtain the collective opinion from experts' opinions.*

Other MCC approaches have been introduced in the specialized literature. In [20] Li et al. pointed out that in real world problems, some experts may not willing to change their opinions too much, because they have a maximum compromise limit. Thus, a novel optimal consensus model with minimum cost that considers experts' compromise was proposed. In this model, weighted dissimilarity is used to measure the cost of reaching the consensus. In [21] two consensus rules were defined, (i) to minimize the distance between the original opinions and adjusted opinions, and (ii) to minimize the number of adjusted values. Gong et al. [19] proposed two different models that find to balance the interests both the moderator and experts. To do that, the theory of primal–dual optimal programming is used. Another MCC model that uses a non-linear least square method to measure the deviation between experts and the collective opinion, was presented in [36]. Li et al. proposed a CRP with minimum cost which considers that the unit adjustment cost of each expert is uncertain and it could be an interval [37]. A new consensus model based on a random distribution considering the probabilistic budget constraint was developed in [38]. Zhang et al. [22] introduced a CRP for GDM problems that deals with incomplete linguistic distribution information. And Cheng et al. [39] studied asymmetric cost situations by using asymmetric cost functions to distinguish the adjustments costs of positive and negative deviations in both directions and proposed three MCC models.

2.3. Comprehensive minimum cost consensus

All MCC approaches introduced in previous section compute the consensus degree by means of the distance between experts and the collective opinion. However, small distances among experts and the collective opinion do not always guarantee to reach an acceptable consensus degree [16]. Therefore, it is necessary to take into account a more CMCC model that considers both the distance of each expert to collective opinion and a minimum agreement among all experts to obtain more acceptable solutions. In order to show this need, let us introduce an example:

Example 1. Let suppose three experts who provide their preferences over an alternative by using a scale from 0 to 1, so $o = \{0.99, 0.45, 0.12\}$. The aim is to achieve a solution in which all the experts agree. We consider $\alpha = 0.85$ a logical consensus threshold to achieve. However, by using the model (M-2), we can only set the minimum distance between the modified experts' preferences, $\bar{o}_k$, and the collective one, $\bar{o}$, through ε. Table 1 shows the consensus level achieved in the group with different values of ε. Note that, with any of them, the consensus threshold $\alpha = 0.85$ is not reached. Therefore, set a minimum distance between experts and collective opinion does not always guarantee to achieve an acceptable solution for the group.

To cope this challenge, Labella et al. have recently presented a novel CMCC model [16] for GDM that is capable of generating high consensual agreement among experts and it can be mathematically described as follows:

$$(\mathbf{M-3})\quad \min \sum_{k=1}^{m} c_k|\bar{o}_k - o_k| \quad \text{s.t.} \begin{cases} \bar{o} = A(\bar{o}_1, \bar{o}_2, \ldots, \bar{o}_m) \\ |\bar{o}_k - \bar{o}| \le \varepsilon,\ k \in \mathbb{I}_m \\ \mathbb{C}(\bar{o}_1, \bar{o}_2, \ldots, \bar{o}_m) \ge \alpha \end{cases}$$

where the function $\mathbb{C} : [a, b]^n \to [0, 1]$ *measures the consensus level among experts and* $\alpha \in [0, 1]$ *is the consensus threshold.*

According to the taxonomy proposed by Palomares et al. [30], the consensus measures used in the first phase of the general scheme of a CRP (see Fig. 1) can be divided into two categories, *consensus measures based on the distance between experts and collective opinion* and *consensus measures based on the distance among the experts*. Labella et al. [16] used the following computations based on Euclidean 1-norm distance to obtain the previous consensus measures:

- Consensus measure based on the distance between experts and the collective opinions given as follows:

$$\mathbb{C}_1(\bar{o}_1, \bar{o}_2, \ldots, \bar{o}_m) = 1 - \sum_{k=1}^{m} w_k|\bar{o}_k - \bar{o}| \quad (1)$$

- Consensus measure based on the distance among experts given as follows:

$$\mathbb{C}_2(\bar{o}_1, \bar{o}_2, \ldots, \bar{o}_m) = 1 - \sum_{k=1}^{m-1}\sum_{l=k+1}^{m} \frac{w_k + w_l}{m-1}|\bar{o}_k - \bar{o}_l| \quad (2)$$

  where $w = (w_1, w_2, \ldots, w_m)$ is the importance of the experts in the DM process.

In our study, we will use the above mentioned consensus measures $\mathbb{C}_1$ and $\mathbb{C}_2$ to obtain the collective opinion from the individual opinions, and we adopt the WA operator, i.e. $A(\bar{o}_1, \bar{o}_2, \ldots, \bar{o}_m) = \sum_{k=1}^{m} w_k\bar{o}_k$ as the aggregation operator. Therefore, by using the consensus measure $\mathbb{C}_1$, the model (M-3) becomes

$$(\mathbf{M-4})\quad \min \sum_{k=1}^{m} c_k|\bar{o}_k - o_k| \quad \text{s.t.} \begin{cases} \bar{o} = \sum_{k=1}^{m} w_k\bar{o}_k \\ |\bar{o}_k - \bar{o}| \le \varepsilon,\ k \in \mathbb{I}_m \\ \sum_{k=1}^{m} w_k|\bar{o}_k - \bar{o}| \le \gamma \end{cases}$$

where $\gamma = 1 - \alpha$.

When $\mathbb{C}_2$ measure is used to compute the consensus degree, the MCC model (M-3) is transformed into the following:

$$(\mathbf{M-5})\quad \min \sum_{k=1}^{m} c_k|\bar{o}_k - o_k| \quad \text{s.t.} \begin{cases} \bar{o} = \sum_{k=1}^{m} w_k\bar{o}_k \\ |\bar{o}_k - \bar{o}| \le \varepsilon,\ k \in \mathbb{I}_m \\ \sum_{k=1}^{m-1}\sum_{l=k+1}^{m} \frac{w_k + w_l}{m-1}|\bar{o}_k - \bar{o}_l| \le \gamma \end{cases}$$

2.4. Fuzzy preference relations and their consistency

In DM, experts usually express their opinions over different alternatives by using preference relations [40], which are pairwise comparisons matrices between alternatives. A preference relation is noted as $P = (p_{ij})_{n\times n}$ where p_{ij} can be interpreted as the preference of the alternative x_i over x_j.

This contribution focuses on one of the most common preference relations used in DM, the FPRs [24] in which the preferences between alternatives are represented by membership degrees:

Definition 2 (*[24]*)**.** A FPR, P, over the set X is defined by a fuzzy set on the product set $X \times X$ and characterized by the membership function $\mu_P : X \times X \to [0, 1]$ and $p_{ij} = \mu_P(x_i, x_j) \in [0, 1]$.

In a FPR, $p_{ij} = 0.5$ indicates indifference between x_i and x_j, $p_{ij} = 1$ indicates x_i is absolutely preferred to x_j and $p_{ij} > 0.5$ indicates x_i is preferred to x_j.

Definition 3 (*[41]*)**.** A FPR, $P = (p_{ij})_{n\times n}$, is said to be additive iff $p_{ij} + p_{ji} = 1\ \forall\, i, j \in \mathbb{I}_n$ and $i \neq j$.

Consistency in FPRs is key to avoid that experts do not provide their opinions randomly and the results obtained are reliable. Interpreting the value $p_{ij} - 0.5$ as a preference intensity, Tanino [41] defined the additive transitive property of a FPR, P, as follows:

$$(p_{ij} - 0.5) + (p_{jt} - 0.5) = p_{it} - 0.5\ \forall\, i, j, t \in \mathbb{I}_n. \quad (3)$$

Based on Eq. (3), we can infer whether or not an additive FPR is consistent.

Definition 4 (*[41]*)**.** An additive FPR, $P = (p_{ij})_{n\times n}$, is said to be consistent iff

$$p_{ij} = p_{it} + p_{tj} - 0.5\ \forall\, i, j, t \in \mathbb{I}_n \quad (4)$$

It is expected that experts provide consistent FPRs based on their knowledge and expertise but the inconsistency in the judgments occurs quite often due to the bounded rational behavior of the experts together with the complexity of the decision scenario, limited knowledge and time-pressure. Thus, to measure the level of consistency in FPRs is essential to obtain reliable results. In a nutshell, the consistency level could be measured by computing the average violation in transitivity for the preferences among the triplet (x_i, x_j, x_t). In this vein, Herrera et al. [42] defined a measure for the consistency level as follows:

Definition 5 (*[42]*)**.** The consistency level of an additive FPR, $P = (p_{ij})_{n\times n}$, is given by

$$CL(P) = 1 - \frac{2}{3n(n-1)(n-2)} \sum_{i\neq j\neq t} |p_{ij} + p_{jt} - p_{it} - 0.5| \quad (5)$$

It is evident that $CL(P) \in [0, 1]$. When $CL(P) = 1$, the additive FPR P is fully consistent. Otherwise, the lower the value of $CL(P)$, the more inconsistent is P. As the permutation over indices makes the identity $a_{ijt} = a_{itj} = a_{jit} = a_{jti} = a_{tij} = a_{tji}$ holds for all i, j, t with $a_{ijt} = |p_{ij} + p_{jt} - p_{it} - 0.5|$, we can re-write Eq. (5) as follows:

$$CL(P) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i<j<t} |p_{ij} + p_{jt} - p_{it} - 0.5| \tag{6}$$

3. Novel comprehensive MCC approaches for LS-GDM

Firstly, this section analyzes and studies the relations between MCC and CMCC models. Afterwards, the non-linear CMCC models are transformed into linear models in order to deal with LS-GDM problems with utility vectors and FPRs. Finally, new linear CMCC models focused on FPRs are extended to consider experts' preferences consistency.

3.1. Relationship between MCC and CMCC models

The MCC model (M-2) and the CMCC model (M-3) are distinguished by the consensus constraints that dictates agreement among the experts but, there is a connection among the optimal consensus cost and, under certain conditions, both models obtain same consensus opinions.

In order to establish a link among MCC and CMCC models, first it is assumed that Λ_1 and Λ_2 are the feasible regions of the models (M-2) and (M-4), respectively. Since, the model (M-4) contains one additional constraint, the feasible region of Λ_2 is a subset of the Λ_1, i.e., $\Lambda_2 \subset \Lambda_1$. This fact relates the models (M-2) and (M-4) with their corresponding objective values.

Theorem 1. *Let z_1^* and z_2^* be the optimal objective values of the models (M-2) and (M-4). Then, $z_2^* \geq z_1^*$. Further, if $\varepsilon \leq \gamma$, then $z_2^* = z_1^*$ and the optimal solution $\{\bar{o}_1^*, \bar{o}_2^*, \dots, \bar{o}_m^*\}$ of (M-2) is also an optimal solution of (M-4).*

Proof. Since, the models (M-2) and (M-4) have the same objective function $\sum_{k=1}^m |\bar{o}_k - o_k|$ and $\Lambda_2 \subset \Lambda_1$, the minimization of their objectives satisfies $\min_{\Lambda_2} \sum_{k=1}^m |\bar{o}_k - o_k| \geq \min_{\Lambda_1} \sum_{k=1}^m |\bar{o}_k - o_k|$, and consequently, $z_2^* \geq z_1^*$.

Let $\{\bar{o}_1^*, \bar{o}_2^*, \dots, \bar{o}_m^*\}$ be the optimal solution with optimal objective value z_1^* of (M-2). Then, we have

$$\sum_{k=1}^m w_k |\bar{o}_k^* - \bar{o}^*| \leq \sum_{k=1}^m w_k \varepsilon = \varepsilon \leq \gamma \tag{7}$$

Clearly, $\{\bar{o}_1^*, \bar{o}_2^*, \dots, \bar{o}_m^*\}$ is a feasible solution of (M-4). In fact, when $\varepsilon \leq \gamma$, the additional consensus constraint in (M-4) becomes redundant and therefore (M-4) is transformed into the MCC model (M-2). Hence, $\{\bar{o}_1^*, \bar{o}_2^*, \dots, \bar{o}_m^*\}$ is the optimal solution of (M-2) and so, $z_2^* = z_1^*$. □

Remark 1. In the same vein of Theorem 1, it is easy to establish that $z_1^* \leq z_3^*$, where z_3^* is the optimal objective value of the CMCC model (M-5).

It is evident from the construction of the CMCC models (M-4) and (M-5) that they only differ in the measurement of the consensus level. This fact leads to explore the connection between the consensus measures used in (M-4) and (M-5). The relation between the consensus measures i.e. Eqs. (1) and (2) is illustrated in the following theorem:

Theorem 2. *For any weighting vector $w = (w_1, w_2, \dots, w_m)$ with $w_k \geq 0$ and $\sum_{k=1}^m w_k = 1$, the consensus measures satisfy*

$$\mathbb{C}_1(o_1, \dots, o_m) \leq \mathbb{C}_2(o_1, \dots, o_m) \tag{8}$$

where $\bar{o} = \sum_{k=1}^m w_k o_k$ and $o_k \in [0, 1]$ for all $k \in \mathbb{I}_m$.

Proof. We note that

$$\begin{aligned}
CL_1(o_1, \dots, o_m) &= \sum_{k=1}^m w_k |o_k - \bar{o}| \\
&= \sum_{k=1}^m w_k |o_k(w_1 + \dots + w_m) - \sum_{l=1}^m w_l o_l| \\
&= \sum_{k=1}^m w_k | \sum_{l=1, l\neq k}^m w_l (o_k - o_l)| \\
&\leq \sum_{k=1}^m w_k \sum_{l=1, l\neq k}^m w_l |o_k - o_l| \\
&\leq \frac{\sum_{k=1}^m \sum_{l=1, t\neq k}^m w_l |o_k - o_l|}{\sum_{k=1}^m \sum_{l=1, l\neq k}^m w_l} \\
&= \sum_{k=1}^{m-1} \sum_{l=k+1}^m \frac{w_k + w_l}{m-1} |o_k - o_l|
\end{aligned}$$

Hence, $\mathbb{C}_1(o_1, \dots, o_m) \leq \mathbb{C}_2(o_1, \dots, o_m)$. □

Based on the above relation between the consensus measures, we can establish the connection between the optimal consensus cost of the CMCC models (M-4) and (M-5) as follows:

Theorem 3. *Let $z^*_{(M-4)}$ and $z^*_{(M-5)}$ be the optimal consensus cost of the CMCC models (M-4) and (M-5), respectively. For any values of the thresholds $\varepsilon > 0$ and $\gamma > 0$, we have, $z^*_{(M-5)} \geq z^*_{(M-4)}$*

3.2. Linear CMCC models with numerical utility vectors

Undoubtedly, the CMCC models (M-4) and (M-5) are non-linear due to the presence of the absolute value constraints and may not be efficient to find the consensus solution for LS-GDM problems. Therefore, it seems necessary to find corresponding equivalent linear programming (LP) models, which can efficiently solve problems with a large number of experts. In the following, we describe the existence of the optimal solution and equivalent LP models for the CMCC models (M-4) and (M-5).

It is almost evident that the CMCC models (M-4) and (M-5) have always feasible solutions. In particular, $\bar{o}_k = \bar{o}, \ \forall k \in \mathbb{I}_m$ is a feasible solution for the models (M-4) and (M-5) for any value of $\varepsilon \geq 0$ and $\gamma \geq 0$. In other words, the same opinion for all the experts (unanimity) provides us a feasible solution.

Proposition 1. *CMCC models (M-4) and (M-5) have optimal solutions for any value of the consensus parameters $\varepsilon \geq 0$ and $\gamma \geq 0$.*

To find the optimal solution of the model (M-4) in an efficient way, we transform the optimization model (M-4) into a LP model as follows:

Theorem 4. *The CMCC model (M-4) can be transformed into an equivalent linear programming model with the help of the following transformations*

$$\bar{o}_k - o_k = u_k \text{ and } |\bar{o}_k - o_k| = v_k, \ \ k \in \mathbb{I}_m, \tag{9}$$

$$\bar{o}_k - \bar{o} = y_k \text{ and } |\bar{o}_k - \bar{o}| = z_k, \ \ k \in \mathbb{I}_m, \tag{10}$$

and it can be put into the following form:

(M − 6)

$$\min \sum_{k=1}^{m} c_k v_k$$

$$s.t. \begin{cases} \bar{o} = \sum_{k=1}^{m} w_k \bar{o}_k & \text{(a)} \\ \bar{o}_k - o_k = u_k, \quad k \in \mathbb{I}_m & \text{(b)} \\ u_k \le v_k, \quad k \in \mathbb{I}_m & \text{(c)} \\ -u_k \le v_k, \quad k \in \mathbb{I}_m & \text{(d)} \\ \bar{o}_k - \bar{o} = y_k, \quad k \in \mathbb{I}_m & \text{(e)} \\ y_k \le z_k, \quad k \in \mathbb{I}_m & \text{(f)} \\ -y_k \le z_k, \quad k \in \mathbb{I}_m & \text{(g)} \\ z_k \le \varepsilon, \quad k \in \mathbb{I}_m & \text{(h)} \\ \sum_{k=1}^{m} w_k z_k \le \gamma. & \text{(i)} \end{cases} \tag{11}$$

Proof. In (M-6), constraints (11)(e)–(11)(h) correspond to the transformation of $|\bar{o}_k - \bar{o}| \le \varepsilon,\ k \in \mathbb{I}_m$ in (M-4). The constraint (11)(i) ensures the consensus condition $\sum_{k=1}^{m} w_k |\bar{o}_k - \bar{o}| \le \gamma$ in (M-4) is satisfied. Now, as $|\bar{o}_k - o_k| = v_k\ \forall\ k$, the objective functions of (M-4) and (M-6) are the same and the constraints (11)(b)–(11)(d) make sure that absolute values property $|u_k| = v_k\ \forall\ k \in \mathbb{I}_m$. Therefore, (M-4) can be transformed into the equivalent LP model (M-6). □

Further, Theorem 4 infers that the optimal solution of the CMCC model (M-4) can be obtained by solving the linear programming model (M-6).

Similarly, the optimal solution of the model (M-5) can be obtained by transforming it into an equivalent LP model as follows:

Theorem 5. *The CMCC model (M-5) can be transformed into an equivalent linear programming model with the help of the following transformations*

$$\bar{o}_k - o_k = u_k \text{ and } |\bar{o}_k - o_k| = v_k,\ k \in \mathbb{I}_m \tag{12}$$

$$\bar{o}_k - \bar{o}_l = y_{kl},\ \text{and } |\bar{o}_k - \bar{o}_l| = z_{kl}\ k \in \mathbb{I}_{m-1},\ l \in \mathbb{I}_m^{k+1} \tag{13}$$

and it can be put into the following form:

(M − 7)

$$min \sum_{k=1}^{m} c_k v_k$$

$$s.t. \begin{cases} \bar{o}_k - \bar{o} \le \varepsilon, k \in \mathbb{I}_m \\ \bar{o} - \bar{o}_k \le \varepsilon, k \in \mathbb{I}_m \\ \bar{o} = \sum_{i=1}^{m} w_k \bar{o}_k \\ \bar{o}_k - o_k = u_k, k \in \mathbb{I}_m \\ u_k \le v_k, k \in \mathbb{I}_m \\ -u_k \le v_k, k \in \mathbb{I}_m \\ \bar{o}_k - \bar{o}_l = y_{kl}, k \in \mathbb{I}_{m-1},\ l \in \mathbb{I}_m^{k+1} \\ y_{kl} \le z_{kl}, k \in \mathbb{I}_{m-1},\ l \in \mathbb{I}_m^{k+1} \\ -y_{kl} \le z_{kl}, k \in \mathbb{I}_{m-1},\ l \in \mathbb{I}_m^{k+1} \\ \sum_{k=1}^{m-1} \sum_{t=k+1}^{m} \frac{w_k + w_l}{m-1} (y_{kl} + z_{kl}) \le \gamma \end{cases}$$

where $\mathbb{I}_m^{k+1} = \{k+1, k+2, \ldots, m\}$ is the restriction on the index set $\mathbb{I}_m$ by considering only positive integers greater than or equal to $k+1$.

Proof. The proof of Theorem 5 follows in the lines of the proof of Theorem 4. □

3.3. Linear CMCC models with FPRs

As we aforementioned, FPRs is one of the structures most commonly used in GDM problems to represent the preferences elicited by experts. For this reason, Labella et al. [16] extended the CMCC models to deal with FPRs:

Definition 6 (*[16]*)**.** Let $P_k = (p_{ij}^k)_{n \times n},\ k \in \mathbb{I}_{m-1}$ be the assessments in the form of FPRs provided by a set of experts $E = \{e_1, \ldots, e_m\}$ over the set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, where p_{ij}^k is the preference of the alternative x_i over x_j given by expert e_k. We further assume that c_k is the cost of moving expert e_k's opinion 1 unit. The CMCC model based on a linear cost function is given as follows:

(M − 8)

$$\min \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_k |p_{ij}^k - \bar{p}_{ij}^k|$$

$$\text{s.t.} \begin{cases} \bar{p}_{ij} = \sum_{k=1}^{m} w_k \bar{p}_{ij}^k,\ i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ |\bar{p}_{ij}^k - \bar{p}_{ij}| \le \varepsilon,\ k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ \mathbb{C}(\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_t) \ge \alpha \end{cases}$$

where $\bar{P}_k = (\bar{p}_{ij}^k),\ k = 1, 2, \ldots, m$ is the adjusted FPRs of the experts, $\bar{P} = (\bar{p}_{ij})$ is the adjusted collective opinion that is obtained by aggregating all the adjusted FPRs of the experts by means of weighted average operator, i.e., $\bar{p}_{ij} = \sum_{k=1}^{t} w_k \bar{p}_{ij}^k$, $\varepsilon \in [0, 1]$ measures the deviation between expert's adjusted preference and collective preference, α dictates the desired consensus level, and $\mathbb{C} : \mathbb{M}_{n\times n}^m \to [0, 1]$ is a consensus measure of the experts' opinions expressed via FPRs where $\mathbb{M}_{n\times n}$ dictates the set of all additive preference relation over the finite set of alternatives $\{x_1, x_2, \ldots, x_n\}$.

Therefore, Labella et al. [16] extended the consensus measures (Eqs. (1) and (2)) to deal with FPRs and proposed the following CMCC models:

- Model based on the consensus measure that considers the distance of each expert to the collective opinion:

(M − 9)

$$\min \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_k |p_{ij}^k - \bar{p}_{ij}^k|$$

$$\text{s.t.} \begin{cases} \bar{p}_{ij} = \sum_{k=1}^{m} w_k \bar{p}_{ij}^k,\ i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ |\bar{p}_{ij}^k - \bar{p}_{ij}| \le \varepsilon,\ k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ l \in \mathbb{I}_n^{i+1} \\ \dfrac{2}{n(n-1)} \displaystyle\sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_k |\bar{p}_{ij}^k - \bar{p}_{ij}| \le \gamma \end{cases}$$

- Model based on the consensus measure that considers the distance among experts:

(M − 10)

$$\min \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_k |p_{ij}^k - \bar{p}_{ij}^k|$$

$$\text{s.t.} \begin{cases} \bar{p}_{ij} = \sum_{k=1}^{m} w_k \bar{p}_{ij}^k,\ i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ |\bar{p}_{ij}^k - \bar{p}_{ij}| \le \varepsilon,\ k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ \frac{2}{n(n-1)} \displaystyle\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \frac{w_k + w_l}{n-1} |\bar{p}_{ij}^k - \bar{p}_{ij}^l| \le \gamma \end{cases}$$

Evidently both CMCC models (M-9) and (M-10) are non-linear in nature and finding optimal solution directly may not be efficient in the large scale scenario. Thus, we transform them into equivalent LP models, which shares the same optimal solutions as non-linear counter parts and that have been described in next two theorems.

Theorem 6. *The CMCC model (M-9) can be transformed into an equivalent linear programming model with the help of the following transformations*

$$p_{ij}^k - \overline{p}_{ij}^k = u_{ij}^k \text{ and } |\overline{p}_{ij}^k - \overline{p}_{ij}| = v_{ij}^k, \tag{14}$$

$$\overline{p}_{ij}^k - \overline{p}_{ij} = y_{ij}^k; \text{ and } |\overline{p}_{ij}^k - \overline{p}_{ij}| = z_{ij}^k, \tag{15}$$

for all $k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1}$ *and it can be put into the following form:*

$(\mathbf{M-11})$

$$\min \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k v_{ij}^k$$

$$\text{s.t.} \begin{cases} \overline{p}_{ij} = \sum_{k=1}^m w_k \overline{p}_{ij}^k,\ i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ p_{ij}^k - \overline{p}_{ij}^k = u_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ -u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij}^k - \overline{p}_{ij} = y_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ y_{ij}^k \le z_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ -y_{ij}^k \le z_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ z_{ij}^k \le \varepsilon, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ \frac{2}{n(n-1)} \sum_{k=1}^m \sum_{i=1}^{n-1} \sum_{j=i+1}^n w_k z_{ij}^k \le \gamma \end{cases}$$

Theorem 7. *The CMCC model (M-10) can be transformed into an equivalent linear programming model with the help of the following transformations*

$$p_{ij}^k - \overline{p}_{ij}^k = u_{ij}^k \text{ and } |\overline{p}_{ij}^k - p_{ij}^k| = v_{ij}^k, \tag{16}$$

$$\overline{p}_{ij}^k - \overline{p}_{ij}^l = y_{ij}^{kl} \text{ and } |\overline{p}_{ij}^k - \overline{p}_{ij}^l| = z_{ij}^{kl}, \tag{17}$$

for all $k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1}$ *and it can be put into the following form:*

$(\mathbf{M-12})$

$$\min \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k v_{ij}^k$$

$$\text{s.t.} \begin{cases} \overline{p}_{ij}^k - \overline{p}_{ij} \le \varepsilon,\ k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij} - \overline{p}_{ij}^k \le \varepsilon,\ k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij} = \sum_{k=1}^m w_k \overline{p}_{ij}^k,\ i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ p_{ij}^k - \overline{p}_{ij}^k = u_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ -u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij}^k - \overline{p}_{ij}^l = y_{ij}^{kl},\ k \in \mathbb{I}_{m-1}, l \in \mathbb{I}_m^{k+1}, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ y_{ij}^{kl} \le z_{ij}^{kl}, k \in \mathbb{I}_{m-1}, l \in \mathbb{I}_m^{k+1}, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ -y_{ij}^{kl} \le z_{ij}^{kl}, k \in \mathbb{I}_{m-1}, l \in \mathbb{I}_m^{k+1}, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ \frac{2}{n(n-1)} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m} \frac{w_k + w_l}{n-1} z_{ij}^{kl} \le \gamma \end{cases}$$

The proof of Theorems 6 and 7 are similar to Theorem 4, thus we omitted here.

3.4. New CMCC models considering consensus and consistency in FPRs

Even though the CMCC models (M-11) and (M-12) could generate the adjusted opinions for any given value of the ε and γ, there is no guarantee that the adjusted FPRs would maintain the sufficient consistency. Although the initial experts' opinions are consistent, it is not sure that the adjusted FPRs and collective opinion are consistent. That may result in unreliable final decision outcomes. Therefore, it seems necessary to study the consistency in the FPRs to overcome this drawback.

The idea is to preserve a certain level of consistency in the adjusted opinions of the experts and collective opinion when we are targeting to achieve a certain level of consensus among experts. The consistency measure of the additive FPRs could allow us to check the consistency level of the experts' opinions. One way to preserve the reasonable consistency levels in the adjusted opinions is to restrict the search space of the adjusted opinions to those FPRs that have the desired levels of consistency set by experts. That can be simply done by including the consistency level constraint in the CMCC models (M-11) and (M-12).

- Model based on the consensus measure that considers the distance of each expert to the collective opinion and consistency in FPRs:

$(\mathbf{M-13})$

$$\min \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k v_{ij}^k$$

$$\text{s.t.} \begin{cases} \overline{p}_{ij} = \sum_{k=1}^m w_k \overline{p}_{ij}^k,\ i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ p_{ij}^k - \overline{p}_{ij}^k = u_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ -u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij}^k - \overline{p}_{ij} = y_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ y_{ij}^k \le z_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ -y_{ij}^k \le z_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ z_{ij}^k \le \varepsilon, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1},\ j \in \mathbb{I}_n^{i+1} \\ \frac{2}{n(n-1)} \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_k z_{ij}^k \le \gamma \\ 1 - \frac{4}{n(n-1)(n-2)} \sum_{i<j<l} |\overline{p}_{ij}^k + \overline{p}_{jl}^k - \overline{p}_{il}^k - 0.5| \ge \delta_k, k \in \mathbb{I}_m \\ 1 - \frac{4}{n(n-1)(n-2)} \sum_{i<j<l} |\overline{p}_{ij} + \overline{p}_{jl} - \overline{p}_{il} - 0.5| \ge \delta \end{cases}$$

where the last two (tenth and eleventh) constraints make sure desired consistency in the adjusted opinions of the each expert and collective opinion, $\delta_k > 0 (k = 1, 2, \ldots, m)$ *is the desired consistency threshold that the expert k seeks to maintain in the CRP to generate meaningful results, and* δ *is the desired consistency threshold in the collective opinion.*

Note that the CMCC model (M-13) with consistency constraints is non-linear for the presence of the new constraints involving absolute values. Transforming the absolute values by introducing new variables and constraints, we can formulate an equivalent LP model to find the optimal solution model (M-13) efficiently and that can be done with the help of following theorem:

Theorem 8. *The CMCC model (M-13) can be transformed into a LP model with the following transformations*

$$\overline{p}_{ij}^k + \overline{p}_{jl}^k - \overline{p}_{il}^k - 0.5 = d_{ijl}^k \text{ and } |\overline{p}_{ij}^k + \overline{p}_{jl}^k - \overline{p}_{il}^k - 0.5| = e_{ijl}^k$$

for all $i < j < l$ (18)

$\overline{p}_{ij} + \overline{p}_{jl} - \overline{p}_{il} - 0.5 = d_{ij}$ *and* $|\overline{p}_{ij} + \overline{p}_{jl} - \overline{p}_{il} - 0.5| = e_{ij}$

for all $i < j < l$ (19)

and it takes into the following form:

(M − 14)

$$\min \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k v_{ij}^k$$

$$s.t. \begin{cases} \overline{p}_{ij} = \sum_{k=1}^{m} w_k \overline{p}_{ij}^k, \ i \in \mathbb{I}_{n-1}, \ j \in \mathbb{I}_n^{i+1} \\ p_{ij}^k - \overline{p}_{ij}^k = u_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, \ j \in \mathbb{I}_n^{i+1} \\ u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, \ j \in \mathbb{I}_n^{i+1} \\ -u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, \ j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij}^k - \overline{p}_{ij} = y_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, \ j \in \mathbb{I}_n^{i+1} \\ y_{ij}^k \le z_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, \ j \in \mathbb{I}_n^{i+1} \\ -y_{ij}^k \le z_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, \ j \in \mathbb{I}_n^{i+1} \\ z_{ij}^k \le \varepsilon, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, \ j \in \mathbb{I}_n^{i+1} \\ \frac{2}{n(n-1)} \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_k z_{ij}^k \le \gamma \\ \overline{p}_{ij}^k + \overline{p}_{jl}^k - \overline{p}_{il}^k - 0.5 = d_{ijl}^k, i<j<l, k \in \mathbb{I}_m \\ d_{ijl}^k \le e_{ijl}^k, i<j<l, k \in \mathbb{I}_m \\ -d_{ijl}^k \le e_{ijl}^k, i<j<l, k \in \mathbb{I}_m \\ \frac{4}{n(n-1)(n-2)} \sum_{i<j<l} e_{ijl}^k \le 1 - \delta_k, k \in \mathbb{I}_m \\ \overline{p}_{ij} + \overline{p}_{jl} - \overline{p}_{il} - 0.5 = d_{ijl}, i,j,l \in \mathbb{I}_n, i<j<l \\ d_{ijl} \le e_{ijl}, i,j,l \in \mathbb{I}_n, i<j<l \\ -d_{ijl} \le e_{ijl}, i,j,l \in \mathbb{I}_n, i<j<l \\ \frac{4}{n(n-1)(n-2)} \sum_{i<j<l} e_{ijl} \le 1 - \delta, \end{cases}$$

- Model based on the consensus measure that considers the distance among experts and consistency in FPRs:

(M − 15)

$$\min \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k v_{ij}^k$$

$$\text{s.t.} \begin{cases} \overline{p}_{ij}^k - \overline{p}_{ij} \le \varepsilon, \ k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij} - \overline{p}_{ij}^k \le \varepsilon, \ k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij} = \sum_{k=1}^{m} w_k \overline{p}_{ij}^k, \ i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ p_{ij}^k - \overline{p}_{ij}^k = u_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ -u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij}^k - \overline{p}_{ij}^l = y_{ij}^{kl}, k \in \mathbb{I}_{m-1}, l \in \mathbb{I}_{m-1}^{k+1}, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ y_{ij}^{kl} \le z_{ij}^{kl}, k \in \mathbb{I}_{m-1}, l \in \mathbb{I}_{m-1}^{k+1}, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ -y_{ij}^{kl} \le z_{ij}^{kl}, k \in \mathbb{I}_{m-1}, l \in \mathbb{I}_{m-1}^{k+1}, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ \frac{2}{n(n-1)} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m} \frac{w_k+w_l}{n-1} z_{ij}^{kl} \le \gamma \\ 1 - \frac{4}{n(n-1)(n-2)} \sum_{i<j<l} |\overline{p}_{ij}^k + \overline{p}_{jl}^k - \overline{p}_{il}^k - 0.5| \ge \delta_k, k \in \mathbb{I}_m \\ 1 - \frac{4}{n(n-1)(n-2)} \sum_{i<j<l} |\overline{p}_{ij} + \overline{p}_{jl} - \overline{p}_{il} - 0.5| \ge \delta \end{cases}$$

Analogously, it can be found the optimal solution of (M-15) with the help of the following theorem:

Theorem 9. *Similar to Theorem 8, the CMCC model (M-15) can be transformed into a LP model as follows:*

(M − 16)

$$\min \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_k v_{ij}^k$$

$$s.t. \begin{cases} \overline{p}_{ij}^k - \overline{p}_{ij} \le \varepsilon, \ k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij} - \overline{p}_{ij}^k \le \varepsilon, \ k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij} = \sum_{k=1}^{m} w_k \overline{p}_{ij}^k, \ i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ p_{ij}^k - \overline{p}_{ij}^k = u_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ -u_{ij}^k \le v_{ij}^k, k \in \mathbb{I}_m, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ \overline{p}_{ij}^k - \overline{p}_{ij}^l = y_{ij}^{kl}, k \in \mathbb{I}_{m-1}, l \in \mathbb{I}_{m-1}^{k+1}, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ y_{ij}^{kl} \le z_{ij}^{kl}, k \in \mathbb{I}_{m-1}, l \in \mathbb{I}_{m-1}^{k+1}, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ -y_{ij}^{kl} \le z_{ij}^{kl}, k \in \mathbb{I}_{m-1}, l \in \mathbb{I}_{m-1}^{k+1}, i \in \mathbb{I}_{n-1}, j \in \mathbb{I}_n^{i+1} \\ \frac{2}{n(n-1)} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{m-1}\sum_{l=k+1}^{m} \frac{w_k+w_l}{n-1} z_{ij}^{kl} \le \gamma \\ \overline{p}_{ij}^k + \overline{p}_{jl}^k - \overline{p}_{il}^k - 0.5 = d_{ijl}^k, i<j<l, k \in \mathbb{I}_m \\ d_{ijl}^k \le e_{ijl}^k, i,j,l \in \mathbb{I}_n, i<j<l, k \in \mathbb{I}_m \\ -d_{ijl}^k \le e_{ijl}^k, i,j,l \in \mathbb{I}_n, i<j<l, k \in \mathbb{I}_m \\ \frac{4}{n(n-1)(n-2)} \sum_{i<j<l} e_{ij}^k \le 1 - \delta_k, i<j<l, k \in \mathbb{I}_m \\ \overline{p}_{ij} + \overline{p}_{jl} - \overline{p}_{il} - 0.5 = d_{ijl}, i,j,l \in \mathbb{I}_n, i<j<l \\ d_{ijl} \le e_{ijl}, i,j,l \in \mathbb{I}_n, i<j<l \\ -d_{ijl} \le e_{ijl}, i,j,l \in \mathbb{I}_n, i<j<l \\ \frac{4}{n(n-1)(n-2)} \sum_{i<j<l} e_{ij} \le 1 - \delta \end{cases}$$

As CMCC models (M-11) and (M-14) differs only in consistency level constraints, the optimal cost of reaching consensus via these models may be connected. In fact, the additional consistency constraints in (M-14) may increase optimal cost of reaching consensus and that fact has been depicted in the following theorem:

Theorem 10. *Let $z^*_{(M-11)}$ and $z^*_{(M-14)}$ be the optimal cost of moving opinions of the experts obtained from CMCC models (M-11) and (M-14) respectively. For any values of the thresholds $\epsilon > 0$ and $\gamma > 0$, we have $z^*_{(M-14)} \ge z^*_{(M-11)}$.*

Remark 2. In the lines of Theorem 10, we can also establish the connection on the optimal costs of moving opinions between CMCC models (M-12) and (M-16) as $z^*_{(M-16)} \ge z^*_{(M-12)}$.

So far, we have proposed several equivalent LP models corresponding to the different CMCC models, which are originally non-linear in nature and we have extended them to deal with FPRs and their consistency. The equivalent LP models are easy to solve in comparison with their non-linear counterparts and thus, we can apply these models in LS-GDM problems to reach consensus with minimal cost of moving experts' opinions. But the complexity of these models remains as concerns when we attempt to find the solution of an LS-GDM problem with hundreds of experts involved in the problem. In the next section, we shed light on this aspect along with tools to solve the proposed LP models.

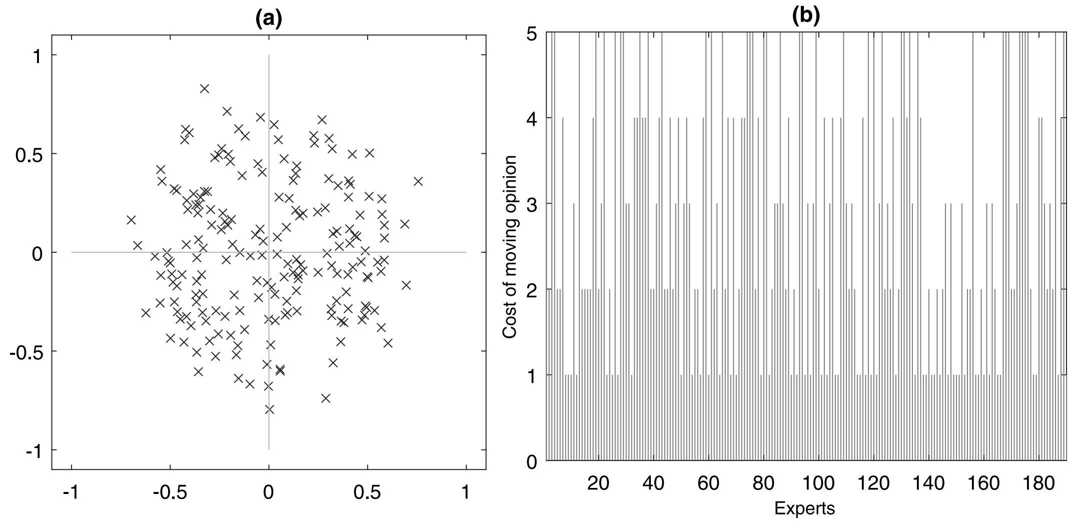

Fig. 2. (a) Visualization of the 190 experts' opinions via multi-dimensional scaling; (b) distribution of the cost of moving opinions of the experts.

4. Illustrative example: comparative and complexity analysis

This section shows an illustrative example to demonstrate the feasible applicability of our proposal in LS-GDM problems and a comparative study with another model with similar characteristics to our proposed CMCC models. Additionally, the complexity of the proposed models considering both their size and run time is also analyzed, but first, we introduce some tools that can be used to solve such models.

Remark 3. Unlike most of proposals in LS-GDM, in this illustrative example we use almost two hundred experts whose opinions have been generated randomly in *Julia* under the condition that their minimum consistency have to be at least 0.8.

4.1. Solver tools

Undoubtedly, LP models are easier to solve than their counterpart equivalent non-linear models. There are well-designed solvers out in the open-source ecosystem that can solve LP models with thousands of variables and constraints along with commercial solvers, like, *CPLEX*[1] and *Gurobi*.[2] Note that Gurobi and CPLEX solvers are available at free of cost to the academic community for the use of only academic purpose. In addition to LP solvers, we require a tool for modeling environments that will allow us to write algebraic mathematical models. A common approach is to create a problem statement in a modeling environment and then pass the problem to a solver. We have followed this architecture to solve the proposed CMCC models for LS-GDM. To do so, we have chosen *JuMP* (Julia for Mathematical Programming), an open-source modeling language, that allows users to define a wide range of optimization problems (linear, mixed-integer, quadratic, conic-quadratic, semi-definite, and non-linear) in a high-level algebraic syntax [43]. As open-source LP solvers, we have chosen Clp[3] and GLPK[4] that is easily interfaced with JuMP at present along with other commercial solvers.[5]

1 https://www.ibm.com/analytics/cplex-optimizer.
2 https://www.gurobi.com/.
3 https://projects.coin-or.org/Clp.
4 https://www.gnu.org/software/glpk/.
5 http://www.juliaopt.org/JuMP.jl/v0.13/.

Table 2
CMCC models optimal cost.

Model	Cost
(M-11)	123.16
(M-12)	239.37
(M-14)	123.55
(M-16)	239.43

4.2. Illustrative example

Let $E = \{e_1, \ldots, e_{190}\}$ be a set of 190 members of the United Nations that attempt to reach a consensus to prioritize a set of climate change policies, $\{x_1, x_2, x_3, x_4\}$. The cost of moving experts' opinions varies for the different member states, as everyone has different economic goals to achieve, which may directly affect the environment policy. Therefore, the persuasion resource required to move opinions for different stakeholders varies quite significantly. The goal for the United Nations is to reach a consensus agreement by deploying minimum resources in the persuasion process for moving opinions of the stakeholders. We mimic the data of the mentioned decision scenario generating the FPRs randomly for each expert under the constraints that the minimum consistency of each generated FPRs should be at least 0.8 and the associated cost of moving opinions is assigned randomly from the scale $\{1, 2, 3, 4, 5\}$. The experts' opinions, along with his/her moving cost, are presented in Fig. 2, and the original data is available in the link https://bit.ly/3gzEa3r.

Due to the large number of stakeholders involved in the GDM problem implies a significant amount of resources, United Nations aims to attain a soft consensus and wish to obtain at least $\alpha = 0.85$ level of consensus agreement. The maximum deviation of any stakeholder from the collective opinion should not exceed $\varepsilon = 0.3$. To achieve the consensus agreement among the stakeholders by expending minimal resources to persuade them, we first employ the proposed models (M-11) and (M-14). The adjusted opinions obtained from these models are reported in Fig. 3(a–b).

In terms of optimal cost (see Table 2) and consistency, the model (M-11) achieves the consensus with an optimal cost of moving opinions 123.16. Although the desired consensus threshold has been achieved, there are 13 experts whose consistency of the adjusted opinions falls short from the initial consistency level, at least 0.8. The reason behind that the model (M-11) does not impose any consistency constraint over the adjusted

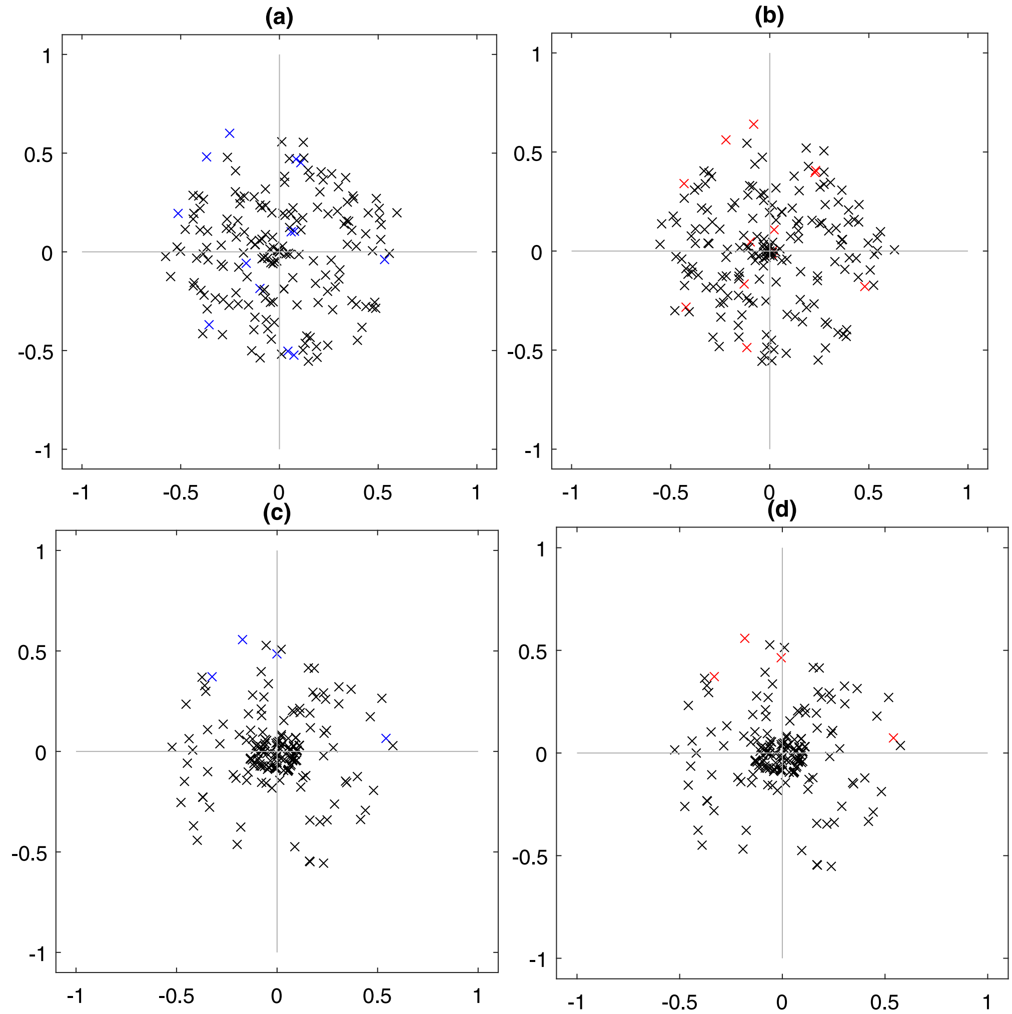

Fig. 3. (a) Visualization of the 190 experts' opinions obtained via model (M-11) where blue color emphasizes that the consistency in the expert's adjusted opinion is below 0.8; (b) Visualization of the 190 experts' consensus opinions obtained via model (M-14) where red color denotes the opinions of those blue colored experts with consistency at least 0.8; (c) Visualization of the 190 experts' consensus opinions obtained via model (M-12) where blue color emphasizes that the consistency in the expert's adjusted opinion is below 0.8; (d) Visualization of the 190 experts' consensus opinions obtained via model (M-16) where red color denotes the opinions of those blue colored experts with consistency at least 0.8.

opinions to preserve a minimal consistency level. On the other hand, in the model (M-14), we impose the condition that the adjusted opinions should have a minimum consistency level of 0.8 and that implies an increment in the optimal cost of moving opinions 123.55. Thus, the employing of the CMCC model (M-14) is much more reasonable to maintain sufficient consistency in the adjusted opinions of the experts.

Next, we solved the LS-GDM problem with the CMCC models (M-12) and (M-16), where the consensus level is computed via the consensus measure based on the distance among experts $\mathbb{C}_2$ and the results are reported in Fig. 3(c–d). In comparison with the model (M-11), the results from the model (M-12) shows that there are only four experts whose consistency level is below 0.8 at the desired consensus agreement among experts. But there is a sharp increase in the optimal cost of moving experts' opinions. In fact, it is almost doubled, 239.37, as it is shown in Table 2. Evidently, we can say that the use of the consensus level, driven from a distance among experts, leads to expending more resources in the persuasion process. To maintain the desired consistency level in the adjusted opinions of the experts under $\mathbb{C}_2$ consensus measure, we employed the CMCC model (M-16) to obtain the consensus opinions with minimal cost of opinions, the results are depicted in Fig. 3(d). We obtained the consensus opinions of the experts with desired consistency level for all the experts with a bit of increment in optimal opinion changing cost 239.43, which supports Remark 2. In a nutshell, it indicates that maintaining experts consistency along with consensus level, demands more resources in the persuasion process.

4.3. Comparative performance analysis

In order to show the advantages and strengths of our proposed linear CMCC models compared to other models, this section introduces a comparative performance analysis. Such analysis consists of the study of different aspects: (i) time complexity, (ii) optimal cost and (iii) consistency level in resulting FPRs. However, not all the consensus models presented in the literature are adequate if we want to make a proper and fair comparison. For this reason, we based our selection on by following:

- Consensus models without feedback mechanism: the proposed CMCC models do not provide a feedback mechanism and experts' preferences are automatically modified. On the other hand, consensus models with feedback process require the participation of experts and depend on them to achieve the consensual solution. Therefore, a comparison with consensus models with feedback would be not fair both in terms of time complexity and general performance.

Table 3
Optimal results from different consensus model.

Model	Time (s)	Cost
Zhang et al. [44]	187.25	194.18
(M-16)	51.6	97.090

- Optimal solution: the proposed CMCC models make use of linear programming models that obtain the optimal solution for a specific LS-GDM problem. Therefore, it seems logical to discard those consensus models that follow an algorithmic process to obtain the solution, because they will not be optimal, in most cases. Thus, we select those models with linear programming approaches.
- Experts' preferences modeled by FPRs: this comparative analysis is focuses on the CMCC models (M-14) and (M-16) thus, the selected models have to deal with FPRs.
- Consistency control in FPRs: the proposed CMCC models guarantee to obtain a solution in which the resulting FPRs have a desired level of consistency. Therefore, we should consider those models that also take into account this aspect.

Based on the previous points, we have selected the model proposed by Zhang et al. in [44] to carry out the comparative performance analysis. This model is based on linear optimization and allows to fix a desired level of consistency for the resulting experts' preferences modeled by FPRs, which makes it an ideal model to compare with our proposal. Due to Zhang et al.'s model computes the consensus level via the consensus measure based on the distance among experts $\mathbb{C}_2$, the comparison will be between such a model and our CMCC model (M-16). Table 3 shows the results related to time and cost obtained from the models for group decision scenario described in Section 4.2.

The results obtained clearly show that the CMCC model (M-16) is better both in terms of cost and time, which means that it is much more effective in addressing the problem of scalability in LS-GDM and is also capable of finding a better optimal solution than the model proposed by Zhang et al. Regarding consistency in the resulting FPRs, both models guarantee a desired minimum level of consistency, in this case 0.8, for all the FPRs.

Remark 4. Note that, since Zhang et al.'s model does not consider the cost of moving opinions, in order to make a proper comparison between both models, we set the cost of changing opinions for all the experts equal to 1. The rest of the parameters are assigned with the same values as defined in Section 4.2.

Remark 5. For the comparison study, Zhang et al.'s model and (M-16) are solved in Julia 1.3.1 on a desktop with Windows 10 Education OS, 3.40 GHz Intel core i7-6700 CPU and 16 GB RAM by invoking the Gurobi 9.0.1 optimizer.

4.4. Complexity of the different CMCC models regarding their size

Typically, the complexity of the LP models depends on the size of the problem, i.e., the number of constraints and variables. The run times or the memory allocation also depend on the size of the problems for fix computational resources. Therefore, we analyze the complexity of the proposed CMCC models dealing with FPRs.

When a group of m experts express their assessments over a set of alternatives $\{x_1, x_2, \dots, x_n\}$ by using FPRs, the size of the problem increases proportionately to corresponding with numerical utility vector and their sizes in terms of constraint and variables depicted in Table 4. The complexity of the models (M-14) and (M-16) increases from the models (M-11) and (M-12) due to impose of non-linear consistency constraints on experts' modified FPRs and collective FPR. In order to linearize the non-linear consistency constraints, we need to introduce additional $\frac{(2m+2)n(n-1)(n-2)}{6}$ continuous variables with $\frac{(m+1)n(n-1)(n-2)}{2} + 2$ linear constraints).

4.5. Complexity of CMCC models regarding their run time

Although a performance metric has been proposed recently to compare different CRP processes based on the minimum cost consensus solution [16], there is no standard GDM dataset for very large-scale instances along with minimum cost solutions to facilitate comparisons with other methods. Such datasets are essential to validate the performance of the new CRPs proposals and facilitate the comparison with other models. Thus, we report a few LS-GDM datasets with their minimum cost consensus solutions obtained via models (M-14) and (M-16) to advocate the future proposals in this direction.

Time is also a vital factor when we attempt to conduct LS-GDM problems in real-time via distributed platform MCC models. Therefore, we analyze the time taken by different solvers for finding optimal consensus opinions employing different CMCC models. To do so, we have considered the LS-GDM scenarios involving the assessment of the set of alternatives $\{x_1, x_2, x_3, x_4\}$ with a different number of experts in the datasets. The experts provide their assessments by using FPRs with the consistency of at least 0.8. The cost of moving experts opinions are the same, i.e., $c_i = 1$ for all the experts in the group. We further assume that all the experts' opinions are equally important in the formation of the group.

With this mentioned set up, we have created the datasets for six LS-GDM scenarios by varying the number of experts in the group $\{50, 100, 150, 200, 250, 300\}$. We will refer the LS-GDM scenario with 50 experts as *FPR50* and the same will follow for the rest. For each instance, we attempt to find the MCC opinions of the experts by solving the CMCC models (M-14) and (M-16), and we have invoked four solvers, namely, Gurobi, Cplex, GLPK and Clp in Julia under same computational configuration to obtain the solutions. It is important to note that we have worked with the default configuration of solvers without any parameters change. The optimal cost of moving opinions obtained by solving corresponding CMCC models (M-14) and (M-16) with different solvers are reported in Table 5 along with the real-time taken by Julia with that solver configuration. Note that we have set the consensus level threshold $\alpha = 0.8$ and $\varepsilon = 0.3$ for the experiments. The datasets, along with the solution obtained by the different solvers and consistency of each experts' preference, are available in https://bit.ly/37SB5av.

From the solvers' perspectives, it is quite evident from Table 5 that all the solvers could able to find the optimal solution for all the instances under CMCC model (M-14) and optimal costs are the same. But the academic version of Gurobi and Cplex turns out to be faster than the open-source solvers GLPK and Clp. On the other hand, when we deploy the CMCC model (M-16) to find the consensus opinions of the experts, there is an exponential increase in the execution time to generate the results and Gurobi outperforms all in terms of execution time. Moreover, the open-source solver GLPK and Clp are unable to produce solutions for the group involving more than 150 experts with a time limit of 1800 seconds.

Table 4
Model size in terms of constraints and variables under FPRs assessments.

Model	No opinions	No of cont. variables	No of constraints
(M-11)	$\frac{mn(n-1)}{2}$	$\frac{(5m+1)n(n-1)}{2}$	$\frac{(7m+1)n(n-1)}{2}+1$
(M-12)	$\frac{mn(n-1)}{2}$	$\frac{(m+1)^2n(n-1)}{2}$	$\frac{(3m^2+7m+2)n(n-1)}{4}+1$
(M-14)	$\frac{mn(n-1)}{2}$	$\frac{(5m+1)n(n-1)}{2}+\frac{(2m+2)n(n-1)(n-2)}{6}$	$\frac{(7m+1)n(n-1)}{2}+\frac{(m+1)n(n-1)(n-2)}{2}+3$
(M-16)	$\frac{mn(n-1)}{2}$	$\frac{(m+1)^2n(n-1)}{2}+\frac{(2m+2)n(n-1)(n-2)}{6}$	$\frac{(3m^2+7m+2)n(n-1)}{4}+\frac{(m+1)n(n-1)(n-2)}{2}+3$

Table 5
Optimal cost and run time for different datasets.

Scenario	Model	Gurobi		Cplex		GLPK		Clp	
		Time (s)	Cost	Time (s)	Cost	Time (s)	Cost	Time (s)	Cost
FPR50	(M-14)	0.0197	8.0860	0.0139	8.0860	0.0867	10.4999	0.0595	8.0860
	(M-16)	0.3829	18.6296	1.0125	18.6296	8.0860	18.6296	6.7076	18.6296
FPR100	(M-14)	0.0414	17.0111	0.0436	17.0111	0.3817	17.0111	0.1929	17.0111
	(M-16)	2.2344	35.1356	7.9614	35.1356	289.9235	35.1356	153.3702	35.1356
FPR150	(M-14)	0.0701	25.9368	0.0806	25.9368	0.8495	25.9368	0.4564	25.9368
	(M-16)	8.0099	50.1374	34.6883	50.1374	–	–	–	–
FPR200	(M-14)	0.1054	31.7640	0.1250	31.7640	1.4985	31.7640	0.6103	31.7640
	(M-16)	23.5157	66.6181	80.6378	66.6181	–	–	–	–
FPR250	(M-14)	0.1421	40.8489	0.1353	40.8489	2.2928	0.8489	0.8759	40.8489
	(M-16)	55.5469	85.1735	187.5703	85.1735	–	–	–	–
FPR300	(M-14)	0.1723	49.9816	0.1959	49.9816	3.5846	49.9816	1.3870	49.9816
	(M-16)	133.4059	100.1862	536.6646	100.1862	–	–	–	–

5. Conclusions

LS-GDM problems are increasingly common and require CRPs to smooth disagreements, which are very common due to the participation of a large number of experts, and reach consensual solutions in which most of the group agrees. However, experts may not be receptive to changing their opinions too much. MCC models allow to obtain a consensual solution by preserving as much as possible the initial experts' opinions but they increase the level of agreement in the group just considering the distance among the experts' preferences and the collective opinions, which may not be enough to guarantee a desired level of consensus. Recently, CMCC models were proposed to overcome the previous drawback considering the computation of the consensus by means of different consensus measures. Nevertheless, these CMCC models are focused on GDM problems with a few numbers of experts, and they may not be efficient to find the consensual solution in LS-GDM problems. Moreover, they do not consider the consistency of the experts' opinions.

This contribution has introduced new CMCC linear models to deal with LS-GDM problems in which experts provide their preferences both numerical assessments and FPRs. Additionally, in the CMCC for FPRs, their consistency has been taken into account in order to obtain reliable results. Finally, the usefulness of the new models has been shown through a LS-GDM case study which studies the complexity of such models and includes a comparative performance analyses to display the advantages of the proposed models. In a nutshell, the comparison indicates that keeping experts consistency along with consensus level requires more resources in the persuasion process.

As future research, we will study the use and performance of different aggregation operators to obtain the collective opinion in the CMCC linear models defined in this proposal.

CRediT authorship contribution statement

Rosa M. Rodríguez: Writing - original draft, Writing - review & editing, Investigation. **Álvaro Labella:** Validation, Investigation, Writing - original draft, Visualization. **Bapi Dutta:** Methodology, Writing - original draft, Software, Writing - review & editing, Formal analysis. **Luis Martínez:** Conceptualization, Formal analysis, Supervision, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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