Information Sciences 512 (2020) 352–368

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Note on entropies of hesitant fuzzy linguistic term sets and their applications

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A R T I C L E I N F O

Article history:
Received 11 January 2019
Revised 23 April 2019
Accepted 8 June 2019
Available online 9 June 2019

Keywords:
Group decision making
Hesitant fuzzy linguistic term set
Entropy
Fuzziness
Hesitancy
Weights

A B S T R A C T

Hesitant fuzzy linguistic term set (HFLTS) is very useful in depicting the situations where people are hesitant to provide their opinions or assessments. In a HFLTS, it should be considered two types of uncertainty, fuzziness and hesitation. This paper is aimed to investigate the problem of how apply different uncertainty facets in different decision making settings. First, a new construction method of a fuzzy entropy for HFLTSs is proposed and it is compared with other methods already introduced in the literatures. Afterwards, these entropy formulas are used to propose two algorithms for deriving the criteria weights and experts weights. Different from the existing applications, it is stressed that in the process of deriving the criteria weights, only the hesitancy of the HFLTS should be considered, while in the process of deriving the experts weights with hesitant fuzzy preference relation information, both the fuzziness and hesitancy of the evaluation information should be involved.

1. Introduction

In multi-criteria decision making (MCDM), many criteria are of qualitative nature, and it is more suitable to evaluate them by using linguistic terms. Hence, it is very natural and important to study the fuzzy linguistic approach [39]. Up to now, many linguistic models have been proposed to extend and improve the fuzzy linguistic approach regarding information modeling and computing processes [25]. Nevertheless, they are limited because the linguistic term set used is defined a priori and experts provide their opinions by using only one linguistic term and sometimes due to the lack of information and time pressure, the use of only one linguistic term it is not enough, since experts can hesitate among several of them [26]. In order to manage these situations, Rodríguez et al. [23,24] presented the concept of Hesitant Fuzzy Linguistic Term Set (HFLTS) motivated by hesitant fuzzy sets [6,12,13,27]. This concept has attracted the attention of many researchers who have introduced different approaches to deal with this type of information. In [20], some arithmetic and geometric aggregation operators to fuse HFLTSs were defined. Wei et al. [32] studied the aggregation process considering the weights of the arguments and proposed two aggregation operators. In [15], some similarity measures to compare HFLTSs were introduced. Different similarity measures based on cosine were proposed in [19] and several ordered weighted distance operators for HFLTS were defined in [36]. Zhao et al. [42] introduced some approaches to develop distance measures for HFLTS based on t-norm

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https://doi.org/10.1016/j.ins.2019.06.018

and t-conorm functions. Zhu and Xu [43] defined some consistency measures for hesitant fuzzy linguistic preference relations (HFLPRs) and proposed a consistency index to fix the consistency thresholds of HFLPRs and measure whether a HFLPR has acceptable consistency. The multiplicative consistency of a HFLPR was defined in [40]. Both approaches [40,43] present the same precondition, the HFLTSs of a HFLPR must have the same number of linguistic terms and use a normalization method to assure that all the HFLTSs have the same number of linguistic terms. To avoid the normalization method and thus information distortion, a new consistency measure is presented in [34]. Several multicriteria decision making methods for HFLTS such as, fuzzy TOPSIS [1], VIKOR [18], a new outranking method [28] and a TODIM method based on a score function [29] have been introduced and used to solve decision making problems.

Entropy is an important notion for measuring uncertainty of fuzzy information. There are many studies on entropy measures for fuzzy sets [3], intuitionistic fuzzy sets [22] and hesitant fuzzy sets [33,35]. Recently, the study of entropy measures for HFLTSs has aroused people's interest. Farhadinia [7] presented some entropy measures based on distance measures and similarity measures for HFLTSs. Farhadinia and Xu [8] created a series of entropies of HFLTSs by using of the arithmetic mean of entropies for single linguistic terms in a HFLTS. Gou et al. [10] defined the entropy measures of a HFLTS based on the similarity degree and its negation. These entropies mainly consider the fuzziness degree of a HFLTS, that is the deviation of the linguistic terms contained in a HFLTS from the fuzziest element, but they can not capture all facets of uncertainty. From our view not only the fuzziness should be considered to measure the entropy of a HFLTS, but also the hesitation of the HFLTS should be included in such a computation.

Motivated by these challenges, Farhadinia and Herrera-Viedma [9] introduced the concept of interval-transformed HFLTSs and derived another class of entropy measures for HFLTSs by describing the fuzzy degree and hesitant degree of some closed subintervals of interval [0,1]. In [9], the fuzziness and hesitation of a HFLTS are not directly described.

Wei et al. [30] directly quantified the uncertainty of an extended HFLTS taking into account the fuzziness and hesitation. In [30], the fuzzy entropies of a HFLTS are constructed by calculating the average values of the linguistic terms in the HFLTS. From a new perspective, this paper constructs some new fuzzy entropies to capture the fuzziness of a HFLTS.

Another main purpose of this paper is to discuss the rational applications of these entropy measures. It is easy to show that different decision-making settings, should consider different aspect of uncertainty of hesitant fuzzy linguistic information, so different types of entropy should be adopted to measure the uncertainty of HFLTSs. For example, if HFLTSs are used to represent the evaluations of an alternative under a criterion and the uncertainty of these evaluations is used to assign the importance weight of the criterion, then only the hesitation entropy should be considered. However, if the HFLTSs are used to represent the pair-wise comparison information of some items, and the uncertainty of this information need to be considered, then the fuzziness and hesitation should all be qualified in calculating the uncertainty of this information (more details are explained in Section 4.2). Therefore, we emphasize some points in which we pay attention to the applications of the proposed entropy measures. The contributions of this study are summarized as follows.

(1) A new construction method of fuzzy entropy measures for HFLTSs is introduced and compared with the existing ones.

(2) Algorithms 1 and II are proposed by applying the proposed entropy formulas to derive the criteria weights and experts weights. We point out that in different decision making problems, different aspects of uncertainty should be considered.

(3) Some distance measures are proposed for HFLTSs. To do so, it is not necessary to add any linguistic term to the shorter HFLTS to have the same number of linguistic terms in the two HFTLS, thus the result is more precise because the information is not bias. These distance formulas and the proposed hesitation entropy are combined to derive the criteria weights in Algorithm 1. The benefit of doing so is that more information (including both the derivations and the uncertainty of evaluation information) under a criterion can be considered to compute its weight.

The rest of this paper is structured as follows: Section 2 introduces some basic concepts about HFLTS and several entropy measures already defined. Section 3 constructs some new fuzzy entropies of HFLTSs which are compared with the existing ones. Section 4 presents several distance measures for HFLTSs and two Algorithms to illustrate the rational applications of entropy measures in assigning the criteria and experts weights. Section 5 gives two illustrative examples to show the applications of Algorithms 1 and 2. Some conclusions are pointed out in Section 6.

2. Preliminaries

This section reviews some concepts about HFLTS and several entropy measures already defined.

2.1. Hesitant fuzzy linguistic term sets

Linguistic scales are the basis of linguistic decision making. The most widely used one is a finite and totally ordered linguistic term set, denoted by $S=\{s_0, s_1, \dots, s_g\}$, with odd cardinality [2,4,5,14]. For example, a set S with seven terms could be given as follows: $S=\{s_0:$ nothing, s_1: very low, s_2: low, s_3: medium, s_4: high, s_5: very high, s_6: perfect$\}$. Moreover, it is usually required that the linguistic term set satisfies the following additional characteristics.

(1) There is a negation operator: $Neg(s_i)=s_{g-i}$, where $g+1$ is the cardinality of the linguistic term set;

(2) The set is ordered: $s_i \le s_j$ if and only if $i \le j$, $\forall s_j \in S$. Therefore, there is a maximization operator: $max(s_i, s_j)=s_i$ if $s_j \le s_i$, $\forall s_j \in S$, and a minimization operator: $min(s_i, s_j)=s_i$ if $s_i \le s_j$.

Frequently experts feel hesitation when they have to provide their opinion or preferences due to the lack of knowledge, information or time pressure. In order to model experts' hesitation in such situations, Rodríguez et al. [23] proposed the notion of HFLTS.

Definition 1 [23]**.** Let $S=\{s_0,s_1,\ldots,s_g\}$ be a linguistic term set. A HFLTS H_S on S is defined as an ordered finite subset of consecutive linguistic terms in S:

$$H_S=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\} \text{ such that } s_{\alpha_k}\in S, k=1,2,\ldots,l, \alpha_{k+1}=\alpha_k+1.$$

Definition 2 [23,32]**.** Let $S=\{s_0,s_1,\ldots,s_g\}$ be a linguistic term set and H_S, H_S^1 and H_S^2 be three HFLTS on S,

(1) $\{s_{g-i}\mid s_i\in H_S\}$ is the negation of H_S, denoted by $Neg(H_S)$;
(2) $\max\{s_i|s_i\in H_S\}$ is the upper bound of H_S, denoted by H_S^+; and $\min\{s_i|s_i\in H_S\}$ is the lower bound of H_S, denoted by H_S^-;
(3) $\{max\{s_i,s_j\}\mid s_i\in H_S^1, s_j\in H_S^2\}$ is the max-union of H_S^1 and H_S^2, denoted by $H_S^1\vee H_S^2$;
(4) $\{min\{s_i,s_j\}\mid s_i\in H_S^1, s_j\in H_S^2\}$ is the min-intersection of H_S^1 and H_S^2, denoted by $H_S^1\wedge H_S^2$.

Example 1. Let $S=\{s_0:$ nothing, $s_1:$ very low, $s_2:$ low , $s_3:$ medium, $s_4:$ high, $s_5:$ very high, $s_6:$ perfect$\}$ be a linguistic term set; and $H_S^1=\{s_2,s_3,s_4\}$ and $H_S^2=\{s_4,s_5\}$ be two HFLTSs on S. By using Definition 2, we have

$$Neg(H_S^1)=\{s_{6-4},s_{6-3},s_{6-2}\}=\{s_2,s_3,s_4\},\quad H_S^{1-}=s_2,\quad H_S^{1+}=s_4,$$

$$H_S^1\vee H_S^2=\{max\{s_2,s_4\},max\{s_2,s_5\},max\{s_3,s_4\},max\{s_3,s_5\},max\{s_4,s_4\},max\{s_4,s_5\}\}=\{s_4,s_5\}$$

and

$$H_S^1\wedge H_S^2=\{min\{s_2,s_4\},min\{s_2,s_5\},min\{s_3,s_4\},min\{s_3,s_5\},min\{s_4,s_4\},min\{s_4,s_5\}\}=\{s_2,s_3,s_4\}.$$

We denote $I(s_\alpha)$ the subscript α of the linguistic term s_α.

Definition 3. Let $H_S=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\}$ be a HFLTS on $S=\{s_0,s_1,\ldots,s_g\}$. Then the averaging value of H_S is defined as,

$$\theta(H_S)=\frac{1}{l}\sum_{i=1}^{l}\alpha_i=\frac{1}{2}(\alpha_1+\alpha_l)=\frac{1}{2}(I(H_S^-)+I(H_S^+)). \quad (1)$$

2.2. Entropy measures for HFLTSs

Several entropy measures have been defined for HFLTSs in [7,10,30]. In this section, we summarize these entropies and analyse their characteristics. In order to make the comparative analysis, we transform the entropy measures defined on the linguistic term set, $S'=\{s_{-\tau},\ldots,s_{-1},s_0,s_1,\ldots,s_\tau\}$, in [7,10] to the ones on the linguistic term set, $S=\{s_0,s_1,\ldots,s_g\}$.

2.2.1. Entropy measures in [7–9]

Based on distance or similarity measure between a HFLTS H_S and the linguistic term $s_{\frac{g}{2}}$ representing the most fuzzy element, the following entropies are defined by Farhadinia in [7].

Definition 4 [7]**.** Let $H_S=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\}$ be a HFLTS defined on the linguistic term set $S=\{s_0,s_1,\ldots,s_g\}$.
(1) The entropy measure based on the generalized distance is defined by,

$$E_{F_1^\lambda}(H_S)=1-2\left(\frac{1}{l}\sum_{i=1}^{l}\left(\frac{|\alpha_i-g/2|}{g}\right)^\lambda\right)^{\frac{1}{\lambda}} \quad (2)$$

being l the cardinality of the HFLTS, H_S, and $\lambda>0$.
(2) The entropy measure based on the generalized similarity is defined as follows,

$$E_{F_2^\lambda}(H_S)=1-\left(\frac{1}{l}\sum_{i=1}^{l}\left(\frac{|\alpha_i-g/2|}{g/2}\right)^\lambda\right)^{\frac{2}{\lambda}} \quad (3)$$

being l the cardinality of the HFLTS, H_S, and $\lambda>0$.

In [8], Farhadinia and Xu defined a series of entropies by using the arithmetic mean of entropies for single linguistic terms. In order to be compared, these entropies are normalized such that the maximal values are 1, which are defined as follows,

$$E_{AM1}(H_S)=\frac{1}{l(\sqrt{2}-1)}\sum_{i=1}^{l}\left\{\sin\frac{\pi\alpha_i}{2g}+\sin\frac{\pi}{2}\left(1-\frac{\alpha_i}{g}\right)-1\right\}, \quad (4)$$

$$E_{AM2}(H_S)=\frac{1}{l\ln 2}\sum_{i=1}^{l}\left\{-\frac{\alpha_i}{g}\ln\frac{\alpha_i}{g}-\left(1-\frac{\alpha_i}{g}\right)\ln\left(1-\frac{\alpha_i}{g}\right)\right\}, \tag{5}$$

$$E_{AM3}(H_S)=\frac{1}{l}\sum_{i=1}^{l}\left\{\frac{\min\{\frac{\alpha_i}{g},1-\frac{\alpha_i}{g}\}}{\max\{\frac{\alpha_i}{g},1-\frac{\alpha_i}{g}\}}\right\}, \tag{6}$$

$$E_{AM4}(H_S)=\frac{1}{l}\sum_{i=1}^{l}\left\{2\left(-\frac{1}{4}+\frac{\alpha_i}{g}\left(1-\frac{\alpha_i}{g}\right)\right)+1\right\}, \tag{7}$$

being π the pie number.

The entropies $E_{F_1^\lambda}$ and $E_{F_2^\lambda}$ consider the deviation between the linguistic terms contained in a HFLTS and the fuzziest linguistic term $s_{\frac{g}{2}}$. The smaller the deviation value is, the bigger the entropy. The entropies E_{AM1}, E_{AM2}, E_{AM3} and E_{AM4} consider the deviation between $\frac{\alpha_i}{g}$ and $1-\frac{\alpha_i}{g}$. Since $\frac{g/2}{g}=1-\frac{g/2}{g}$, the entropy of the HFLTS $\{s_{\frac{g}{2}}\}$ is the biggest.

The above entropy measures only consider the fuzziness of a HFLTS. In order to take into account both the fuzziness and the hesitancy, Farhadinia and Herrera-Viedma [9] introduced the concept of interval-transformed HFLTSs and derived another class of entropy measures for HFLTSs by describing the fuzziness and hesitancy of some closed subintervals of interval [0,1].

Definition 5 [9]**.** Let $H_S=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\}$ be a HFLTS defined on the linguistic term set $S=\{s_0,s_1,\ldots,s_g\}$. Then the interval-transformed HFLTS $\mathbb{H}_S$ associated with H_S is defined in terms of the closed intervals by

$$\mathbb{H}_S=\left\{\left[\frac{\alpha_j}{g},\frac{\alpha_{l-j+1}}{g}\right]|j=1,2,\ldots,\left[\frac{l}{2}\right]\right\} \tag{8}$$

being $[\frac{l}{2}]$ the smallest integer no smaller than $\frac{l}{2}$.

Let $I([0,1])$ denote the set of all closed subintervals of [0,1], then an interval-transformed HFLTS can be regarded as the set of some elements in $I([0,1])$, which is also expressed by

$$\mathbb{H}_S=\left\{\mathbb{H}_S(j)=[\mathbb{H}_S^L(j),\mathbb{H}_S^U(j)]|j=1,2,\ldots,\left[\frac{l}{2}\right]\right\}. \tag{9}$$

By considering the uncertainty of intervals $[\mathbb{H}_S^L(j),\mathbb{H}_S^U(j)]$, entropy measures for interval-transformed HFLTSs are constructed as follows,

$$E_{IT1}(\mathbb{H}_S)=\frac{1}{[\frac{l}{2}]}\sum_{i=1}^{[\frac{l}{2}]}\frac{(1-|\mathbb{H}_S^U(j)+\mathbb{H}_S^L(j)-1|)(\mathbb{H}_S^U(j)-\mathbb{H}_S^L(j)+1)}{2}, \tag{10}$$

$$E_{IT2}(\mathbb{H}_S)=\frac{1}{[\frac{l}{2}]}\sum_{i=1}^{[\frac{l}{2}]}\frac{(1-|\mathbb{H}_S^U(j)+\mathbb{H}_S^L(j)-1|)+(\mathbb{H}_S^U(j)-\mathbb{H}_S^L(j))}{2}, \tag{11}$$

$$E_{IT3}(\mathbb{H}_S)=\frac{1}{[\frac{l}{2}]}\sum_{i=1}^{[\frac{l}{2}]}\frac{\left(1-sin\left(\frac{\pi}{2}|\mathbb{H}_S^U(j)+\mathbb{H}_S^L(j)-1|\right)\right)+sin\frac{\pi}{2}(\mathbb{H}_S^U(j)-\mathbb{H}_S^L(j))}{2}. \tag{12}$$

Let $\triangle_j=|\mathbb{H}_S^U(j)+\mathbb{H}_S^L(j)-1|$ and $\nabla_j=\mathbb{H}_S^U(j)-\mathbb{H}_S^L(j)$. $\triangle_j$ can be seen as the distance between the averaging values of $[\mathbb{H}_S^L(j),\mathbb{H}_S^U(j)]$ and that of [0,1], and ∇_j states the deviation value of $[\mathbb{H}_S^L(j),\mathbb{H}_S^U(j)]$. E_{IT1}, E_{IT2} and E_{IT3} are monotonically decreasing with respect to $\triangle_j$ and ∇_j, respectively.

2.2.2. Entropy measures in [10]

Gou et al. [10] proposed the following axiomatic definition and entropy formulas for HFLTSs.

Definition 6 [10]**.** Let $H_S=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\}$ be a HFLTS on a linguistic term set, $S=\{s_0,s_1,\ldots,s_g\}$, and $\mathbb{H}(S)$ be the set of all the HFLTSs. The entropy of H_S is a real-valued function $E:\ \mathbb{H}(S)\to[0,1]$, satisfying the following axiomatic requirements:

- (E1) $E(H_S)=0$, if and only if $H_S=\{s_0\}$ or $H_S=\{s_g\}$;
- (E2) $E(H_S)=1$, if and only if $\alpha_i+\alpha_{l-i+1}=g$ for $i=1,2,\ldots,l$;
- (E3) For a HFLTS, $H_S^1=\{s_{\beta_1},s_{\beta_2}\ldots,s_{\beta_l}\}$, having the same length l with that of H_S, $E({H_S}^1)\geq E(H_S)$, if $\beta_i\leq\alpha_i$ for $\alpha_i+\alpha_{l-i+1}\leqslant g$, or $\beta_i\geq\alpha_i$ for $\alpha_i+\alpha_{l-i+1}\geqslant g$.
- (E4) $E(H_S)=E(Neg(H_S))$.

Based on Definition 6, the entropies of H_S are defined as follows,

$$E_{G_1}(H_S)=\frac{1}{l(\sqrt{2}-1)}\sum_{i=1}^{l}\left\{\sin\frac{\pi(\alpha_i+\alpha_{l-i+1})}{4g}+\sin\pi\left(\frac{1}{2}-\frac{\alpha_i+\alpha_{l-i+1}}{4g}\right)-1\right\}, \tag{13}$$

$$E_{G_2}(H_S)=\frac{1}{l(\sqrt{2}-1)}\sum_{i=1}^{l}\left\{\cos\frac{\pi(\alpha_i+\alpha_{l-i+1})}{4g}+\cos\pi\left(\frac{1}{2}-\frac{\alpha_i+\alpha_{l-i+1}}{4g}\right)-1\right\} \tag{14}$$

and

$$E_{G_3}(H_S)=-\frac{1}{l\ln 2}\sum_{i=1}^{l}\left\{\frac{\alpha_i+\alpha_{l-i+1}}{2g}\ln\frac{\alpha_i+\alpha_{l-i+1}}{2g}+\left(1-\frac{\alpha_i+\alpha_{l-i+1}}{2g}\right)\ln\left(1-\frac{\alpha_i+\alpha_{l-i+1}}{2g}\right)\right\}. \tag{15}$$

The entropy measures E_{G_1}, E_{G_2} and E_{G_3} of a HFLTS H_S are defined based on the similarity degree of H_S and $Neg(H_S)$ and have the following property.

Property 1. Let $H_S^1=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\}$ and $H_S^2=\{s_{\beta_1},s_{\beta_2},\ldots,s_{\beta_t}\}$ be two HFLTSs on the linguistic term set $S=\{s_0,s_1,\ldots,s_g\}$. If H_S^1 and H_S^2, or H_S^1 and $Neg(H_S^2)$ have the same averaging value, then $E_{G_i}(H_S^1)=E_{G_i}(H_S^2), i=1,2,3$.

Proof. Since $\alpha_i=\alpha_{i-1}+1, i=2,3,\ldots,l$, $\theta(H_S^1)=\frac{1}{l}\sum_{i=1}^{l}\alpha_i=\frac{1}{2}(\alpha_i+\alpha_{l-i+1})$, $i=1,2,\ldots,l$. From $\theta(H_S^1)=\theta(H_S^2)$, we have

$$\frac{1}{2}(\alpha_i+\alpha_{l-i+1})=\frac{1}{2}(\beta_k+\beta_{t-k+1}),\quad i=1,2,\ldots,l,\quad k=1,2,\ldots,t.$$

Thus, $E_{G_i}(H_S^1)=E_{G_i}(H_S^2), i=1,2,3$.

If $\theta(H_S^1)=\theta(Neg(H_S^2))$, then $E_{G_i}(H_S^1)=E_{G_i}(Neg(H_S^2))$. Also from $E_{G_i}(H_S^2)=E_{G_i}(Neg(H_S^2))$, thus $E_{G_i}(H_S^1)=E_{G_i}(H_S^2), i=1,2,3$. □

2.2.3. Entropy measures in [30]

Wei et al. [30] pointed out the necessity of considering not only the fuzziness, but also the hesitation of a HFLTS which is reflected by the deviation degree of the linguistic terms present in a HFLTS. Combining both characteristic (fuzziness and hesitation), a comprehensive entropy for Extended HFLTS was introduced in [30]. These axiomatical definitions of entropy measures are as follows.

Definition 7. Let $H_S=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\}$ be a HFLTS on $S=\{s_0,s_1,\ldots,s_g\}$. The deviation function value of a HFLTS H_S is defined by

$$\eta(H_S)=\frac{2}{l(l-1)}\sum_{i=1}^{l-1}\sum_{j=i+1}^{l}(\alpha_j-\alpha_i)=\frac{l+1}{3},\quad l\geq 2. \tag{16}$$

If $H_S=\{s_{\alpha_1}\}$, then we set $\eta(H_S)=0$.

Definition 8. Let $H_S=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\}$ be a HFLTS on the linguistic term set S, $S=\{s_0,s_1,\ldots,s_g\}$, and $\mathbb{H}(S)$ be the set of all the HFLTSs. Let E_f: $\mathbb{H}(S)\rightarrow[0,1]$ be a mapping, E_f is the fuzzy entropy of a HFLTS, if E_f satisfies the following axiomatic requirements:

(F1) $E_f(H_S)=0$, if and only if $H_S=\{s_0\}$ or $H_S=\{s_g\}$;
(F2) $E_f(H_S)=1$, if and only if $H_S=\{s_{\frac{g}{2}}\}$;
(F3) Suppose that H_S^1 and H_S^2 be two HFLTSs with the same number of linguistic terms. If $H_S^{1+}\leqslant H_S^{2+}\leqslant s_{\frac{g}{2}}$ or $H_S^{1-}\geqslant H_S^{2-}\geqslant s_{\frac{g}{2}}$, then $E_f(H_S^1)\leqslant E_f(H_S^2)$;
(F4) $E_f(H_S)=E_f(Neg(H_S))$, where $Neg(H_S)=\{s_{g-\alpha_i}|i=1,2,\ldots,l\}$.

Definition 9. Let $H_S=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\}$ be a HFLTS on a linguistic term set, $S=\{s_0,s_1,\ldots,s_g\}$, and $\mathbb{H}(S)$ be the set of all the HFLTSs. Let E_h: $\mathbb{H}(S)\rightarrow[0,1]$ be a mapping, E_h is the hesitant entropy of a HFLTS, if E_h satisfies the following axiomatic requirements:

(H1) $E_h(H_S)=0$, if and only if $H_S=\{s_i\}(i=0,1,\ldots,g)$;
(H2) $E_h(H_S)=1$, if and only if $H_S=\{s_0,s_1,\ldots,s_g\}$;
(H3) $E_h(H_S^1)\leqslant E_h(H_S^2)$, if $\eta(H_S^1)\leqslant\eta(H_S^2)$;
(H4) $E_h(H_S)=E_h(Neg(H_S))$.

Definition 10. Let $H_S=\{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_l}\}$ be a HFLTS on a linguistic term set, $S=\{s_0,s_1,\ldots,s_g\}$. Let $E_c:\mathbb{H}(S)\rightarrow[0,1]$ be a mapping, E_c is the comprehensive entropy of a HFLTS, if it satisfies the following statements:

(E1) $E_c(H_S)=0$, if and only if $H_S=\{s_0\}$, $H_S=\{s_g\}$;

(E2) $E_c(H_S) = 1$, if and only if $H_S = \{s_{\frac{g}{2}}\}$;
(E3) Let H_S^1 and H_S^2 be two HFLTSs. If $E_f(H_S^1) \leqslant E_f(H_S^2)$ and $E_h(H_S^1) \leqslant E_h(H_S^2)$, then $E_c(H_S^1) \leqslant E_c(H_S^2)$;
(E4) $E_c(H_S) = E_c(Neg(H_S))$;

Taking into account the construction methods of fuzzy entropy measures proposed in [30], the fuzzy entropy formulas of a HFLTS are obtained as follows,

$$E_{f1}^{wrl}(H_S) = \frac{1}{l}\sum_{i=1}^{l}\frac{4\alpha_i}{g}\left(1-\frac{\alpha_i}{g}\right), \tag{17}$$

$$E_{f2}^{wrl}(H_S) = \frac{1}{l}\sum_{i=1}^{l}\frac{1}{\log 2}\left[-\frac{\alpha_i}{g}\log\left(\frac{\alpha_i}{g}\right) - \left(1-\frac{\alpha_i}{g}\right)\log\left(1-\frac{\alpha_i}{g}\right)\right], \tag{18}$$

$$E_{f3}^{wrl}(H_S) = \frac{1}{l}\sum_{i=1}^{l}\frac{\frac{\alpha_i}{g}e^{\left(1-\frac{\alpha_i}{g}\right)} + \left(1-\frac{\alpha_i}{g}\right)e^{\frac{\alpha_i}{g}} - 1}{e^{\frac{1}{2}} - 1}. \tag{19}$$

Since the deviation function value $\eta(H_S)$ is equal to $\frac{l+1}{3}$ $(l \geq 2)$, the deviation function reaches the maximum value $\frac{g+2}{3}$ when $H_S = \{s_0, s_1, \ldots, s_g\}$. According to the axiomatic requirement (H_2): $E_h(H_S) = 1$, if and only if $H_S = \{s_0, s_1, \ldots, s_g\}$, the hesitant entropy of H_S can be revised by

$$E_h(H_S) = f\left(\frac{3}{g+2}\eta(H_S)\right) = f\left(\frac{l+1}{g+2}\right), \quad l \geq 2, \tag{20}$$

where the function f: [0, 1] → [0, 1] is a BUM function that satisfies the following two conditions:

(1) $f(0) = 0,\ f(1) = 1$;
(2) $f(x)$ is strictly monotone increasing on [0,1].

When $l = 1$, $E_h(H_S) = 0$ from the axiomatic requirement $(H1)$.

Note that if the function $f(x)$ in $E_h(H_S)$ defined by Eq. (20) is changed, a series of entropy measures for HFLTSs can be obtained.

For example, we can adopt $f(x) = x^{\alpha}$ with $\alpha > 0$. Thus, for the HFLTS $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$, the hesitant entropy can be calculated by:

$$E_{h\alpha}(H_S) = \left(\frac{3}{g+2}\eta(H_S)\right)^{\alpha} = \left(\frac{l+1}{g+2}\right)^{\alpha}, \quad \alpha > 0,\ l \geq 2. \tag{21}$$

From the axiomatical requirements (E1)–(E4), the comprehensive entropy of H_S can be obtained by

$$E_{c\beta}(H_S) = \frac{E_f + \beta E_h}{1 + \beta E_h}, \beta \in [0, 1]. \tag{22}$$

Example 2. Let $S = \{s_0$: nothing, s_1: very low, s_2: low, s_3: medium, s_4: high, s_5: very high, s_6: perfect} be a linguistic term set. $H_S^1 = \{s_0\}$, $H_S^2 = \{s_8\}$, $H_S^3 = \{s_0, s_1, s_2\}$ and $H_S^4 = \{s_4, s_5, s_6, s_7, s_8\}$ be four HFLTSs on S.

For HFLTSs H_S^1 and H_S^2, $E_{fi}^{wrl}(H_S^1) = E_h(H_S^1) = 0$ and $E_{fi}^{wrl}(H_S^2) = E_h(H_S^2) = 0$, so their comprehensive entropies are also equal to 0.

For HFLTSs H_S^3 and H_S^4, we can calculate their fuzzy entropies by E_{f3}^{wrl} and their hesitation entropies with $\alpha = 1$ as:

$$E_{f3}^{wrl}(H_S^3) = 0.403,\ E_{h1}(H_S^3) = 0.4,\ E_{f3}^{wrl}(H_S^4) = 0.630,\ E_{h1}(H_S^4) = 0.6.$$

Thus their comprehensive entropies can be obtained as $E_{c1}(H_S^3) = 0.574, E_{c1}(H_S^4) = 0.769$. The fuzzy entropies of H_S^3 and H_S^4 are approximately equal to their hesitant entropies, respectively. The comprehensive entropies of H_S^3 and H_S^4 are greater than their fuzzy entropies and hesitation entropies. So the hesitation or fuzziness of H_S^3 and H_S^4 enhances their uncertainty, which is consistent with intuition.

The above fuzzy entropies are constructed by calculating the average values of the fuzzy entropies of the linguistic terms contained in a HFLTS. Motivated by the construction of entropy measures for hesitant fuzzy sets in [41], some new fuzzy entropies are proposed to capture the fuzziness of a HFLTS.

3. Some new fuzzy entropies for HFLTSs

This section proposes a new construction method of fuzzy entropy for HFLTSs considering a new perspective. A comparison with some existing approaches are carried to show the validity and effectiveness of our proposal.

3.1. A new construction method of fuzzy entropies

Theorem 1. *Let* R: $[0, 1]^2 \to [0, 1]$ *be a mapping satisfying the following properties:*

(1) $R(x, y) = 0$ *if and only if* $x = y = 0$ *or* $x = y = 1$;
(2) $R(x, y) = 1$ *if and only if* $x = y = \frac{1}{2}$;
(3) *If* $x_1 \le x_2 \le 0.5$, $y_1 \le y_2 \le 0.5$ *or* $x_1 \ge x_2 \ge 0.5$, $y_1 \ge y_2 \ge 0.5$, *then* $R(x_1, y_1) \le R(x_2, y_2)$;
(4) $R(x, y) = R(1 - y, 1 - x)$ *for any* $x, y \in [0, 1]$.

And let $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ *be a HFLTS on the linguistic term set* $S = \{s_0, s_1, \ldots, s_g\}$. *Then a fuzzy entropy measure for* H_S *is as follows,*

$$E_f(H_S) = \frac{2}{l(l+1)} \sum_{i=1}^{l} \sum_{j \geqslant i}^{l} R\left(\frac{\alpha_i}{g}, \frac{\alpha_j}{g}\right). \tag{23}$$

Proof. Since $R(0, 0) = R(1, 1) = 0$ and $R(\frac{1}{2}, \frac{1}{2}) = 1$, we have $E_f(H_S) = 0$ for $H_S = \{s_0\}$ or $H_S = \{s_g\}$, and $E_f(H_S) = 1$ for $H_S = \{s_{\frac{g}{2}}\}$.

On the other direction, we note that there are $\frac{l(l+1)}{2}$ summands on the right of Eq. (23). If $E_f(H_S) = 0$ for $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$, since $R(x, y) = 0$ only for $x = y = 0$ or $x = y = 1$, we have $l = 1$, and $H_S = \{s_0\}$ or $H_S = \{s_g\}$. If $E_f(H_S) = 1$ for $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$, then $R(\frac{\alpha_i}{g}, \frac{\alpha_j}{g}) = 1$ for any $l \ge j \ge i \ge 1$. Since $R(x, y) = 1$ only for $x = y = \frac{1}{2}$, we have $l = 1$ and $H_S = \{s_{\frac{g}{2}}\}$. Hence we have checked (F1) and (F2) for E_f.

Let $H_S^1 = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ and $H_S^2 = \{s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_l}\}$ be two HFLTSs. Taking into account the conditions $H_S^{1+} \leqslant H_S^{2+} \leqslant s_{\frac{g}{2}}$ or $H_S^{1-} \geqslant H_S^{2-} \geqslant s_{\frac{g}{2}}$, and $\alpha_i = \alpha_{i-1} + 1$, $\beta_i = \beta_{i-1} + 1 (i = 2, \ldots, l)$, we have $s_{\alpha_i} \leqslant s_{\beta_i} \leqslant s_{\frac{g}{2}}$ for $i = 1, 2, \ldots, l$ or $s_{\alpha_i} \geqslant s_{\beta_i} \geqslant s_{\frac{g}{2}}$ for $i = 1, 2, \ldots, l$. And from the condition 3) of $R(x, y)$, it is obtained that $R(\frac{\alpha_i}{g}, \frac{\alpha_j}{g}) \leqslant R(\frac{\beta_i}{g}, \frac{\beta_j}{g})$ for any $l \ge j \ge i \ge 1$. Thus $E_f(H_S^1) \leqslant E_f(H_S^2)$, and (F3) has been checked.

Since $Neg(H_S) = \{s_{g-\alpha_l}, s_{g-\alpha_{l-1}}, \ldots, s_{g-\alpha_1}\}$ for $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$, $E_f(Neg(H_S)) = \frac{2}{l(l+1)} \sum_{i=1}^{l} \sum_{j \geqslant i}^{l} R(\frac{g-\alpha_j}{g}, \frac{g-\alpha_i}{g})$. From the condition 4) of $R(x, y)$, $R(\frac{g-\alpha_j}{g}, \frac{g-\alpha_i}{g}) = R(\frac{\alpha_i}{g}, \frac{\alpha_j}{g})$. Thus, $E_f(Neg(H_S)) = \frac{2}{l(l+1)} \sum_{i=1}^{l} \sum_{j \geqslant i}^{l} R(\frac{\alpha_i}{g}, \frac{\alpha_j}{g}) = E_f(H_S)$.

In conclusion E_f defined by Eq. (23) is a fuzzy entropy of HFLTSs. □

Zhao et al. in Ref. [41] gave the construction method of the function $R(x, y)$ and the following theorem.

Theorem 2. *Let the function* φ: $[0, 1] \to [0, 1]$ *a mapping that satisfies the following two conditions:*

(1) $\varphi(x) = 0$ *if and only if* $x = 0$; $\varphi(x) = 1$ *if and only if* $x = 0.75$;
(2) $\varphi(x)$ *is strictly monotone increasing on [0,0.75] and strictly monotone increasing on [0.75,1].*

If the function R: $[0, 1]^2 \to [0, 1]$ *is defined by* $R(x, y) = \varphi(1 - xy)\varphi(x + y - xy)$, *then* $R(x, y)$ *satisfies the four conditions from Theorem 1.*

Theorem 2 can be directly obtained from the proof process in Ref. [41].
Considering Theorems 1 and 2, the following theorem can be obtained.

Theorem 3. *Let* $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ *be a HFLTS on the linguistic term set* $S = \{s_0, s_1, \ldots, s_g\}$, *suppose*

$$E_f(H_S) = \frac{2}{l(l+1)} \sum_{i=1}^{l} \sum_{j \geqslant i}^{l} \varphi\left(1 - \frac{\alpha_i \alpha_j}{g^2}\right) \varphi\left(\frac{\alpha_i}{g} + \frac{\alpha_j}{g} - \frac{\alpha_i \alpha_j}{g^2}\right) \tag{24}$$

and the function φ: $[0, 1] \to [0, 1]$ *satisfies the two conditions from Theorem 2. Then* E_f *is the fuzzy entropy of a HFLTS.*

From Theorem 3, we can construct different entropy formulas by using different functions φ, selected in [41]. For example, if $\varphi(t) = 1 - (\frac{1}{3}|4t - 3|)^{\gamma}$ with $\gamma \ge 1$, then φ satisfies the two conditions in Theorem 2. From Theorem 3, we obtain the following fuzzy entropy of H_S:

$$E_f^{\gamma}(H_S) = \frac{2}{l(l+1)} \sum_{i=1}^{l} \sum_{j \geqslant i}^{l} \left(1 - \left(\frac{1}{3}\left|\frac{4\alpha_i \alpha_j}{g^2} - 1\right|\right)^{\gamma}\right) \left(1 - \left(\frac{1}{3}\left|\frac{4\alpha_i}{g} - \frac{4\alpha_i \alpha_j}{g^2} + \frac{4\alpha_j}{g} - 3\right|\right)^{\gamma}\right). \tag{25}$$

3.2. A comparative study of entropy measures

An example is solved by using the entropy measures revised in Section 2.2 and the proposed ones.

Table 1
The entropies of $H_S^i (i = 1, 2, \ldots, 9)$ by different entropy measures.

	H_S^1	H_S^2	H_S^3	H_S^4	H_S^5	H_S^6	H_S^7	H_S^8	H_S^9
$\eta(H_S)$	1	0	1.333	0	0	1.333	0	1.333	2
$E_f^1(H_S)$	0.109	0.215	0.214	0.437	0.694	0.838	1	0.704	0.459
$E_f^2(H_S)$	0.249	0.476	0.444	0.775	0.944	0.977	1	0.921	0.702
$E_f^3(H_S)$	0.353	0.654	0.578	0.913	0.990	0.996	1	0.977	0.805
$E_{f1}^{wrl}(H_S)$	0.219	0.438	0.396	0.750	0.938	0.958	1	0.896	0.625
$E_{f2}^{wrl}(H_S)$	0.272	0.544	0.452	0.811	0.954	0.970	1	0.922	0.662
$E_{f3}^{wrl}(H_S)$	0.225	0.499	0.403	0.759	0.940	0.960	1	0.900	0.630
$E_{h1}(H_S)$	0.300	0	0.400	0	0	0.400	0	0.400	0.600
$E_{c0.6} = f(E_f^1, E_{h1})$	0.245	0.215	0.366	0.437	0.694	0.869	1	0.761	0.602
$E_{c0.6} = f(E_f^2, E_{h1})$	0.364	0.476	0.552	0.775	0.944	0.981	1	0.936	0.781
$E_{c0.6} = f(E_f^3, E_{h1})$	0.452	0.654	0.660	0.913	0.990	0.997	1	0.981	0.857
$E_{c0.6} = f(E_{f1}^{wrl}, E_{h1})$	0.338	0.438	0.513	0.750	0.938	0.966	1	0.916	0.724
$E_{c0.6} = f(E_{f2}^{wrl}, E_{h1})$	0.383	0.544	0.558	0.811	0.954	0.976	1	0.937	0.751
$E_{c0.6} = f(E_{f3}^{wrl}, E_{h1})$	0.343	0.499	0.518	0.759	0.940	0.968	1	0.919	0.728
$E_{F_1^1}(H_S)$	0.125	**0.250**	**0.250**	0.5	0.750	0.833	1	0.75	0.500
$E_{F_2^1}(H_S)$	0.234	**0.438**	**0.438**	0.750	0.938	0.917	1	0.938	0.750
$E_{AM1}(H_S)$	0.212	0.425	0.388	0.740	0.934	0.956	1	0.892	0.620
$E_{AM2}(H_S)$	0.272	0.544	0.452	0.811	0.954	0.970	1	0.922	0.662
$E_{AM3}(H_S)$	0.071	0.143	0.159	0.333	0.600	0.733	1	0.644	0.415
$E_{AM4}(H_S)$	0.609	0.719	0.698	0.875	0.969	0.979	1	0.948	0.813
$E_{IT1}(H_S)$	0.070	0.125	0.141	0.250	0.375	0.563	**0.5**	0.422	0.313
$E_{IT2}(H_S)$	0.125	0.125	0.188	0.250	0.375	0.563	**0.5**	0.438	0.375
$E_{IT3}(H_S)$	0.107	0.038	0.134	0.146	0.309	0.596	**0.5**	0.404	0.328
$E_{G_1}(H_S)$	0.225	**0.425**	**0.425**	0.740	0.934	**1**	**1**	0.934	0.740
$E_{G_2}(H_S)$	0.225	**0.425**	**0.425**	0.740	0.934	**1**	**1**	0.954	0.740
$E_{G_3}(H_S)$	0.337	**0.544**	**0.544**	0.811	0.954	**1**	**1**	0.954	0.811

Example 3. Let $S = \{s_0:$ nothing, s_1: very low, s_2: low, s_3: slightly low, s_4: medium, s_5: slightly high, s_6: high, s_7: very high, s_8: perfect$\}$ be a linguistic term set and let

$$H_S^1 = \{s_0, s_1\},\ H_S^2 = \{s_1\},\ H_S^3 = \{s_0, s_1, s_2\},\ H_S^4 = \{s_2\}, H_S^5 = \{s_3\},$$

$$H_S^6 = \{s_3, s_4, s_5\}, H_S^7 = \{s_4\}, H_S^8 = \{s_4, s_5, s_6\}, H_S^9 = \{s_4, s_5, s_6, s_7, s_8\}$$

be nine HFLTSs on S.

The entropies of the above HFLTSs are calculated by the proposed entropy measures E_f^{γ} (with $\gamma = 1, 2, 3$), $E_{fi}^{wrl} (i = 1, 2, 3)$, $E_{h\alpha}$ (with $\alpha = 1$) and $E_{c\beta}$ (with $\beta = 0.6$), respectively. The entropy measures $E_{F_1^{\lambda}}$, $E_{F_2^{\lambda}}$ (with $\lambda = 1$) introduced by Farhadinia, E_{AMi}, $(i = 1, 2, 3, 4)$ by Farhadinia and Xu, $E_{ITi} (i = 1, 2, 3)$ by Farhadinia and Herrera-Viedma, and $E_{Gi} (i = 1, 2, 3)$ by Gou et al. are also computed. The results are shown in Table 1.

Analysing the above entropy formulas and results in Table 1, we can provide the following conclusions:

(1) These entropy formulas are proposed from different points of view. Farhadinia [7] introduced the entropy measures $E_{F_1^1}$ and $E_{F_2^1}$ for HFLTSs based on the distances or similarity degrees between the linguistic terms in a HFLTS H_S and the linguistic term $s_{\frac{g}{2}}$ representing the evaluation "medium" or "equal". Farhadinia and Xu [8] proposed the entropy measures E_{AM1}, E_{AM2}, E_{AM3} and E_{AM4} which are designed by using of the arithmetic mean of the counterparts proposed for linguistic terms. Farhadinia and Herrere-Viedma [9] introduced the concept of interval-transformed HFLTS and derived another class of entropy measures E_{IT1}, E_{IT2} and E_{IT3} for HFLTSs satisfying different axiomatic requirements. Gou et al. [10] defined the entropy measures E_{G_1}, E_{G_2} and E_{G_3} of a HFLTS H_S based on the similarity degree of H_S and $Neg(H_S)$.

The entropy measures $E_{F_1^{\lambda}}$, $E_{F_2^{\lambda}}$, $E_{AMi} (i = 1, 2, 3, 4)$ and $E_{Gi} (i = 1, 2, 3)$ mainly consider the fuzziness of HFLTSs, while E_{ITi}, $(i = 1, 2, 3)$ try to consider both the fuzziness and hesitation of HFLTSs, but these uncertainty are indirectly reflected by means of interval-transformed HFLTSs. Wei et al. [30] directly qualified the uncertainty of a HFLTS from the fuzziness and hesitation through fuzzy entropies E_{f1}^{wrl}, E_{f2}^{wrl}, E_{f3}^{wrl} and hesitant entropy E_h, respectively, and defined the comprehensive entropy $f(E_{fi}^{wrl}, E_h)$ combining both of them. In this paper, the proposed entropy measures $f(E_f^{\gamma}, E_h)$ also directly describe both type of uncertainty.

The characteristics of different entropy measures are concluded in Table 2.

(2) According to different entropy measures, we can ranking the nine HFLTSs. The results are showed in Table 3.

Table 2
The characteristics of different entropy measures.

	E_f^γ	E_{fi}^{wrl}	$E_{h\alpha}$	$f(E_f^\gamma, E_{h\alpha})$	$f(E_{fi}^{wrl}, E_{h\alpha})$	$E_{F_i^\lambda}$	E_{AMi}	E_{ITi}	E_{G_i}
hesitation			√	√	√			√	
fuzziness	√	√		√	√	√	√	√	√

Table 3
The rankings of nine HFLTSs corresponding to different entropy measures.

Entropy measures	The ranking of the nine HFLTSs
E_{G_i}	$H_S^1 < H_S^3 = H_S^2 < H_S^9 = H_S^4 < H_S^8 = H_S^5 < H_S^6 = H_S^7$
$E_{F_i^\lambda}$	$H_S^1 < H_S^3 = H_S^2 < H_S^9 = H_S^4 < H_S^6 < H_S^8 = H_S^5 < H_S^7$
E_{AM3}	$H_S^1 < H_S^2 < H_S^3 < H_S^4 < H_S^9 < H_S^5 < H_S^8 < H_S^6 < H_S^7$
E_f^1	$H_S^1 < H_S^3 < H_S^2 < H_S^4 < H_S^9 < H_S^5 < H_S^8 < H_S^6 < H_S^7$
$E_f^2, E_f^3, E_{fi}^{wrl}, E_{AM1}, E_{AM2}, E_{AM4}$	$H_S^1 < H_S^3 < H_S^2 < H_S^9 < H_S^4 < H_S^8 < H_S^5 < H_S^6 < H_S^7$
$f(E_f^1, E_{h1})$	$H_S^1 < H_S^2 < H_S^3 < H_S^4 < H_S^9 < H_S^5 < H_S^8 < H_S^6 < H_S^7$
$f(E_{fi}^{wrl}, E_{h1}), f(E_f^\gamma, E_{h1}), \gamma = 2, 3$	$H_S^1 < H_S^2 < H_S^3 < H_S^4 < H_S^9 < H_S^8 < H_S^5 < H_S^6 < H_S^7$
E_{IT1}	$H_S^1 < H_S^2 < H_S^3 < H_S^4 < H_S^9 < H_S^5 < H_S^8 < H_S^7 < H_S^6$
E_{IT2}	$H_S^1 = H_S^2 < H_S^3 < H_S^4 < H_S^9 = H_S^5 < H_S^8 < H_S^7 < H_S^6$
E_{IT3}	$H_S^2 < H_S^1 < H_S^3 < H_S^4 < H_S^5 < H_S^9 < H_S^8 < H_S^7 < H_S^6$

We know that $E_f^\gamma, E_{fi}^{wrl}(i = 1, 2, 3)$, $E_{F_i^\lambda}(i = 1, 2)$, $E_{AMi}(i = 1, 2, 3, 4)$ and $E_{Gi}(i = 1, 2, 3)$ consider the fuzziness of HFLTSs. From Table 3, it is showed that the ranking results of the nine HFLTSs are exactly the same corresponding to E_f^2, E_f^3, E_{fi}^{wrl}, E_{AM1}, E_{AM2} and E_{AM4}, while the ranking results are slightly different on some HFLTSs according to others.

$f(E_f^\gamma, E_h)$, $f(E_{fi}^{wrl}, E_h)$ and $E_{ITi}(i = 1, 2, 3)$ consider both the fuzziness and hesitation. From Table 3, we can see the ranking results of these nine HFLTSs are the same corresponding to the comprehensive entropies $f(E_{fi}^{wrl}, E_{h1})(i = 1, 2, 3)$ and $f(E_f^\gamma, E_{h1})(\gamma = 2, 3)$, which are different from that for E_{ITi}. The ranking of HFLTSs H_S^5, H_S^6, H_S^7 and H_S^8 is $H_S^5 < H_S^8 < H_S^7 < H_S^6$ for E_{IT1}, E_{IT2}, while it is $H_S^8 < H_S^5 < H_S^6 < H_S^7$ for $f(E_{fi}^{wrl}, E_{h1})$ and $f(E_f^\gamma, E_{h1})(\gamma = 2, 3)$. The result lies in the different axiomatic requirements of these entropy measures. The entropy measures $f(E_{fi}^{wrl}, E_{h1})$, $f(E_f^\gamma, E_{h1})$ are defined on the base of the requirement that the entropy of $\{s_{\frac{g}{2}}\}$ is 1, and $\{s_{\frac{g}{2}}\}$ is the most uncertain element, while the entropy measures E_{ITi} are defined based on the axiomatical requirement: $E(H_S) = 1$ if and only if $\mathbb{H}_S = \{\mathbb{H}_S(j) = [0, 1] | j = 1, 2, \ldots, [\frac{l}{2}]\}$. It is found that the HFLTS corresponding interval-transformed HFLTS $\{\mathbb{H}_S(j) = [0, 1] | j = 1, 2, \ldots, [\frac{l}{2}]\}$ do not exist. In [9], authors consider that $\{s_0, s_1, \ldots, s_g\}$ is the most uncertain HFLTS, but its corresponding interval-transformed HFLTS is $\mathbb{H}_S = \{[0, 1], [\frac{1}{g}, \frac{g-1}{g}], [\frac{2}{g}, \frac{g-2}{g}], \ldots, [\frac{1}{2}, \frac{1}{2}]\}$.

(3) The entropy measures $E_{G_i}(i = 1, 2, 3)$ and $E_{F_i^1}(i = 1, 2)$ can not discriminate the uncertainty of H_S^2 and H_S^3. In Property 1, we have proved that for two HFLTSs, if they have the same averaging value, the entropies of the two HFLTSs are equal by E_{G_1}, E_{G_2} and E_{G_3}. Since $\theta(H_S^2) = \theta(H_S^3)$ in Example 3, it can be seen from Table 3 that $E_{G_i}(H_S^2) = E_{G_i}(H_S^3)(i = 1, 2, 3)$. On the other hand, the distance and the similar degree of H_S^2 and s_4 are equal to that of H_S^3 and s_4, so the numerical examples in bold type reflect that $E_{F_i^1}(H_S^2) = E_{F_i^1}(H_S^3)(i = 1, 2)$.

Nevertheless, we can see that although H_S^2 and H_S^3 have the same averaging values, their deviation values are different, thus the uncertain degrees should be different. Using the proposed entropy measures E_f^γ (with $\gamma = 1, 2, 3$), $E_{fi}^{wrl}(i = 1, 2, 3)$, $E_{h\alpha}$ (with $\alpha = 1$) and $E_{c\beta}$ (with $\beta = 0.6$), we obtain that the fuzzy entropy of H_S^2 is greater than H_S^3, and the hesitation entropy of H_S^2 is zero being less than that of H_S^3. Thus, the result of the comprehensive entropy is $E_{c0.6}(H_S^2) < E_{c0.6}(H_S^3)$.

(4) The entropy measures $E_{G_i}(i = 1, 2, 3)$ can not distinguish the uncertainty of all the HFLTSs satisfying $\alpha_i + \alpha_{l-i+1} = g$ for $i = 1, 2, \ldots, l$. The fuzzy entropy measures $E_f^\gamma(\gamma = 1, 2, 3)$, $E_{fi}^{wrl}(i = 1, 2, 3)$ and the entropy measures $E_{F_i^1}(i = 1, 2)$ are defined based on the requirement that $s_{\frac{g}{2}}$ is the most fuzziness element, while the entropy measures $E_{G_i}(i = 1, 2, 3)$ are defined based the axiomatical requirement: $E(H_S) = 1$ if and only if $\alpha_i + \alpha_{l-i+1} = g$ for $i = 1, 2, \ldots, l$. So, for HFLTSs H_S^6 and H_S^7, $E_{G_i}(H_S^6) = E_{G_i}(H_S^7) = 1(i = 1, 2, 3)$. The uncertainty of H_S^6 and H_S^7 can not be distinguished by the entropy measures $E_{G_i}(i = 1, 2, 3)$.

However, using the proposed entropy measures, we get that fuzzy entropy of H_S^7 is greater than that of H_S^6, and the hesitation entropy of H_S^7 is less than that of H_S^6; the combined result of these two types of entropies is $E_{c0.6}(H_S^7) > E_{c0.6}(H_S^6)$. The same results are obtained by the entropy measures $E_{F_i^1}(i = 1, 2)$, $E_{F_i^1}(H_S^7) > E_{F_i^1}(H_S^8)(i = 1, 2)$.

4. Applications of the entropy measures

As mentioned in [23,43], a HFLTS can be used as either an evaluation of an alternative under some criteria or as a preference degree for comparing two items. This section focuses on applying the proposed entropy formulas in decision making to derive the criteria weights and the experts weights expressing their pair-wise preferences by HFLTSs. The difference regarding previous applications, it is that we should consider the different aspect of the uncertainty of HFLTSs (fuzziness and hesitation). In the process of deriving the criteria weights, we should only consider the hesitancy of a HFLTS and in the process of computing the experts weights, we should consider both the fuzziness and the hesitancy of the decision making information.

Before introducing the applications of the proposed entropy measures, we will introduce several distance measures for HFLTS, because they will be used in the computation of the criteria weights.

4.1. Distance measures for HFLTSs

Liao et al. [17] introduced the axiomatic definition of the distance measure and some distance formulas for HFLTSs. To do so, the two HFLTSs must have the same length (same number of linguistic terms), thus it is necessary to add linguistic terms to the HFLTS with less linguistic terms until having the same number of linguistic terms. This process called normalization, provokes a bias of the results. In order to obtain more precise results, we propose new distance formulas that do not need to extend the shorter HFLTS. These new distance measures are based on the well-known Hamming distance, Euclidean distance and Hausdorff metric.

Definition 11. Let H_S^1 and H_S^2 be two HFLTSs on the linguistic term set S. A normalized Hamming distance measure for HFLTSs is defined as

$$d_1(H_S^1, H_S^2) = \frac{|I(H_S^{1-}) - I(H_S^{2-})| + |I(H_S^{1+}) - I(H_S^{2+})| + |\theta(H_S^1) - \theta(H_S^2)|}{3g}, \tag{26}$$

a normalized Euclidean distance measure for HFLTSs is defined as

$$d_2(H_S^1, H_S^2) = \left(\frac{(I(H_S^{1-}) - I(H_S^{2-}))^2 + (I(H_S^{1+}) - I(H_S^{2+}))^2 + (\theta(H_S^1) - \theta(H_S^2))^2}{3g^2}\right)^{\frac{1}{2}}, \tag{27}$$

and a normalized Hausdorff distance measure for HFLTS is defined as

$$d_3(H_S^1, H_S^2) = \frac{1}{g}\max\{|I(H_S^{1-}) - I(H_S^{2-})|, |I(H_S^{1+}) - I(H_S^{2+})|, |\theta(H_S^1) - \theta(H_S^2)|\}, \tag{28}$$

where g is determined by the linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$.

We can easily prove that the above three distance measures satisfy the following properties.

Theorem 4. *The distance measures $d_i (i = 1, 2, 3)$ satisfy the following properties:*

(1) $0 \leqslant d_i(H_S^1, H_S^2) \leqslant 1$;
(2) $d_i(H_S^1, H_S^2) = 0$ *if and only if* $H_S^1 = H_S^2$;
(3) $d_i(H_S^1, H_S^2) = d_i(H_S^2, H_S^1)$;
(4) $d_i(H_S^1, H_S^2) \leqslant d_i(H_S^1, H_S^3) + d_i(H_S^2, H_S^3)$;
(5) *If* $H_S^{1-} \leqslant H_S^{2-} \leqslant H_S^{3-}$ *and* $H_S^{1+} \leqslant H_S^{2+} \leqslant H_S^{3+}$, *then* $d_i(H_S^1, H_S^2) \leqslant d_i(H_S^1, H_S^3)$ *and* $d_i(H_S^2, H_S^3) \leqslant d_i(H_S^1, H_S^3)$.
(6) $d_1(H_S^1, H_S^2) \leqslant d_2(H_S^1, H_S^2) \leqslant d_3(H_S^1, H_S^2)$.

Example 4. Let $S = \{s_0$: nothings_1: very low, s_2: low, s_3: medium, s_4: high, s_5: very high, s_6: perfect$\}$ be a linguistic term set; $H_S^1 = \{s_0, s_1, s_2, s_3\}$, $H_S^2 = \{s_2, s_3\}$, $H_S^3 = \{s_2\}$ and $H_S^4 = \{s_3, s_4\}$ be four HFLTSs on S. Then, by Eqs. (26)–(28), we obtain

$$d_1(H_S^1, H_S^4) = \frac{1}{3} \approx 0.333, \quad d_1(H_S^1, H_S^3) = \frac{7}{36} \approx 0.194, \quad d_1(H_S^1, H_S^2) = \frac{1}{6} \approx 0.167;$$

$$d_2(H_S^1, H_S^4) = \left(\frac{7}{54}\right)^{\frac{1}{2}} \approx 0.360, \quad d_2(H_S^1, H_S^3) = \left(\frac{21}{432}\right)^{\frac{1}{2}} \approx 0.221, \quad d_2(H_S^1, H_S^2) = \left(\frac{5}{108}\right)^{\frac{1}{2}} \approx 0.215;$$

$$d_3(H_S^1, H_S^4) = \frac{1}{2} = 0.5, \quad d_3(H_S^1, H_S^3) = \frac{1}{3} \approx 0.333, \quad d_3(H_S^1, H_S^2) = \frac{1}{3} \approx 0.333.$$

Thus we have $d_i(H_S^1, H_S^4) > d_i(H_S^1, H_S^3) > d_i(H_S^1, H_S^2) (i = 1, 2)$, which is consistent with our intuition. By using d_3, the result is $d_3(H_S^1, H_S^4) > d_3(H_S^1, H_S^3) = d_3(H_S^1, H_S^2)$. The discrimination capacity of d_3 is lower than d_1 and d_2. In the following application, we adopt the distance measures d_1 or d_2.

In [16,37,38], the Hausdorff distances for hesitant fuzzy sets are defined based on the forward and backward Hausdorff metric. Motivated by this idea, another Hausdorff distance is defined for HFLTSs that do not need to add linguistic terms to the HFLTS with less linguistic terms.

Definition 12. Let $H_S^1 = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ and $H_S^2 = \{s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_t}\}$ be two HFLTSs on a linguistic term set, $S = \{s_0, s_1, \ldots, s_g\}$. Then the Hausdorff distance of H_S^1 and H_S^2 is defined as

$$d_4(H_S^1, H_S^2) = \max\left\{\max_{1\le i\le l}\min_{1\le k\le t}\left\{\left|\frac{\alpha_i}{g} - \frac{\beta_k}{g}\right|\right\}, \max_{1\le i\le t}\min_{1\le k\le l}\left\{\left|\frac{\alpha_k}{g} - \frac{\beta_i}{g}\right|\right\}\right\}. \tag{29}$$

4.2. Assigning the criteria weights

A multi-criteria linguistic group decision-making problem can be described as follows. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives, and $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set. By using the linguistic term set S, t evaluators or experts $d_1, d_2, \ldots, d_t$ provide evaluations over alternatives x_i $(i = 1, 2, \ldots, n)$ under criteria c_j $(j = 1, 2, \ldots, m)$. Suppose that the evaluation information of the kth expert is represented by a hesitant fuzzy linguistic decision matrix $(H_S^{ij(k)})_{n\times m}$, denoted by R_k, where each $H_S^{ij(k)}$ is a HFLTS or a single linguistic term (which can be regarded as a special HFLTS) in S, and represents the linguistic assessment provided by the expert d_k for the alternative x_i with respect to the criterion c_j. Experts' goal is to obtain the ranking of the alternatives.

Let $H_S^{ij(1)}, H_S^{ij(2)}, \ldots, H_S^{ij(t)}$ be the evaluations provided by t experts and H_S^{ij} the HFLTS that represents the collective evaluation of t experts for the alternative x_i under the criterion c_j. In this problem we will use the HLWA operator [32] to aggregate the evaluations, but any other aggregation operator could be used. The collective decision matrix obtained is represented by $R = (H_S^{ij})_{n\times m}$.

In order to rank the alternatives, we should consider the relative importance of the criteria. Since the information about criteria weights is completely unknown, we propose a method that considers both the deviation degrees and the uncertainty degrees of evaluations to derive the criteria weights according to hesitant fuzzy linguistic information.

The main idea is as follows. On one hand, the bigger the deviation degree under a criterion, the more important the criterion acts for the overall aggregation values. So a bigger weight should be assigned toward this criterion. On the other hand, we usually expect that the total uncertainty degree of the assessments is as small as possible. Thus, the smaller the total uncertainty degree under a criterion, the more important the criterion acts for the overall aggregation values.

According to the above analysis, a method to determine the criteria from the collective decision matrix $R = (H_S^{ij})_{n\times m}$ is proposed.

Algorithm 1. Step 1. Based on Eqs. (26) or (27), calculate the distance between H_S^{ij} and H_S^{lj}, which is denoted by $d(H_S^{ij}, H_S^{lj})$. Afterwards, calculate the weights ω_j^1 of the criteria C_j by

$$\omega_j^1 = \frac{\sum_{i=1}^{n-1}\sum_{l=i+1}^{n} d(H_S^{ij}, H_S^{lj})}{\sum_{j=1}^{m}\sum_{i=1}^{n-1}\sum_{l=i+1}^{n} d(H_S^{ij}, H_S^{lj})}, \quad j = 1, 2, \ldots, m. \tag{30}$$

If $\sum_{i=1}^{n-1}\sum_{l=i+1}^{n} d(H_S^{ij}, H_S^{lj}) = 0$ for any j, then we set $\omega_j^1 = \frac{1}{m}(j = 1, 2, \ldots, m)$.

Step 2. Based on Eq. (21), compute the hesitant entropy of H_S^{ij}, which is denoted by $E_{h\alpha}^{ij}$ and calculate the weights ω_j^2 of criteria C_j by

$$\omega_j^2 = \frac{n - \sum_{i=1}^{n} E_{h\alpha}^{ij}}{nm - \sum_{j=1}^{m}\sum_{i=1}^{n} E_{h\alpha}^{ij}}, \quad j = 1, 2, \ldots, m. \tag{31}$$

Obviously, if $\sum_{i=1}^{n} E_{h\alpha}^{ij} = 0$ for all j, then $\omega_j^2 = \frac{1}{m}(j = 1, 2, \ldots, m)$.

Step 3. Integrate the weights ω_j^1 and ω_j^2 into the weights ω_j of the criteria C_j

$$\omega_j = \lambda\omega_j^1 + (1-\lambda)\omega_j^2, \quad \lambda \in [0, 1], \quad j = 1, 2, \ldots, m. \tag{32}$$

The structure of Algorithm 1 is depicted in Fig. 1.

Remark In the application of assigning the weights of criteria, we only adopt the hesitant entropy E_h. The reason is illustrated by the following example. Suppose experts evaluate the safety and the comfort of three cars by using the linguistic term set $S = \{s_0$: nothing; s_1 : very poor; s_2 : poor; s_3 : medium; s_4 : good; s_5 : very good; s_6 : perfect$\}$. The ratings of the three alternatives with respect to the safety are s_3, s_3, s_3, and the ratings with respect to the comfort are s_6, s_6, s_6, respectively. If we consider the fuzziness of evaluation information to derive the criteria weights, then according to the weighted

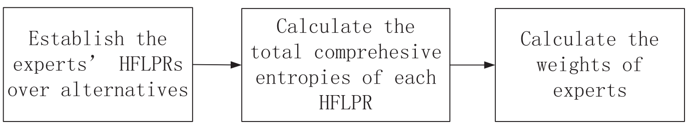

Fig. 1. Structure of Algorithm 1.

method introduced in [7], the weight of the criterion safety is 1 and the weight of the comfort is 0, since the entropy of s_3 is 1, and the entropy of s_6 is 0. It is easy to see that these results are not reasonable. That is because in this setting the fuzziness of evaluation information should not be considered. The evaluations s_3 and s_6 only represent the levels under the two criteria of the cars. Therefore, we do not say that the evaluation s_3 is the fuzziest linguistic term in this setting and only the hesitation of evaluation information should be considered. For the above example, if we only consider the hesitancy of evaluation information as in Algorithm 1, then equal weights should be assigned toward the two criteria.

Fuzziness is considered only when we measure the uncertainty of the evaluation involving comparison of two items. For example, if we compare the safety of the two cars on the linguistic term set $S = \{s_0 : \text{very poor}, s_1 : \text{poor}, s_2 : \text{a little poor}, s_3 : \text{equal}\ , s_4 : \text{a little good}, s_5 : \text{good}, s_6 : \text{very good}\}$, one think that the car A is “equal” to the car B with respect to safety, the other thinks that the car A is “very good ” than the car B with respect to safety, then the evaluation “equal” is more fuzzy than the information “very good ”. In this setting, both fuzziness and hesitation should be considered for these linguistic evaluations, which elicits the following application of the proposed entropies.

4.3. Assigning the experts weights

In this section, the entropy measures of HFLTSs are applied to determine the experts weights for group decision making problems based on hesitant fuzzy linguistic preference relations.

4.3.1. Hesitant fuzzy linguistic preference relations

During the decision making process, an expert usually provides his/her preferences over a set of alternatives, $X = \{x_1, x_2, \ldots, x_n\}$ by using a linguistic preference relation $R = (r_{ij})_{n\times n}$ and a linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$. The element $r_{ij} \in S$ represents the preference degree for the alternative x_i over x_j. Especially, $r_{ij} = s_{\frac{g}{2}}$ indicates indifference between x_i and x_j, $r_{ij} > s_{\frac{g}{2}}$ implies that x_i is preferred to x_j, while $r_{ij} < s_{\frac{g}{2}}$ shows that x_j is preferred to x_i. In those situations where the experts provide pair-wise comparisons of alternatives by using linguistic expressions and hesitate to elicit their preferences, Zhu and Xu [43] presented the following concept of hesitant fuzzy linguistic preference relations (HFLPRs):

Definition 13 [43]**.** Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set, and $R = (H_S^{ij})_{n\times n}$ be a matrix, where $H_S^{ij} = \{s_{\alpha_l^{ij}} | l = 1, 2, \ldots, \#H_S^{ij}\}$ is a HFLTS on S indicating that the degree to which alternative x_i is preferred to x_j. R is called a hesitant fuzzy linguistic preference relation (HFLPR) if it satisfies the following conditions: $H_S^{ii} = \{s_{\frac{g}{2}}\}$ and H_S^{ji} is the negation of H_S^{ij} for any $i, j = 1, 2, \ldots, n$.

Next we discuss how to assign the experts weights taking into account the experts' preferences over the alternatives, when the information regarding the experts weights is completed unknown.

4.3.2. A method to determine the experts weights

The group decision making problem considered in this section can be described as follows: let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of alternatives, $E = \{e_1, e_2, \ldots, e_m\}$ be the set of experts. The expert e_k provides his/her preferences for each pair of alternatives by using the linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$, and constructs a HFLPR $R^{(k)} = (H_S^{ij^{(k)}})_{n\times n}$, where $H_S^{ij^{(k)}} = \{s^{(k)}_{\alpha_l^{ij}} | l = 1, 2, \ldots, \#H_S^{ij^{(k)}}\}$, $H_S^{ii^{(k)}} = \{s_{\frac{g}{2}}\}$, and $H_S^{ji^{(k)}} = Neg(H_S^{ji^{(k)}})$ for all $i, j = 1, 2, \ldots, n$.

For the group decision making problem based on HFLPRs, the experts weights need to be incorporated into each individual. In the case that the weight information of the experts is completely unknown, we derive the experts weights from their corresponding HFLPRs, since the HFLPRs which express the experts' preference information may reflect their real understandings toward the alternatives. Next, an approach is proposed to derive the experts weights based on the corresponding HFLPRs.

Each HFLPR $R^{(k)} = (H_S^{ij^k})$ $(k = 1, 2, \ldots, m)$ is constructed by $n \times n$ HFLTSs. The element $H_S^{ij^k}$ represents the comparison information of alternative x_i and x_j. The closer $H_S^{ij^k}$ is to $s_{\frac{g}{2}}$, the more fuzzy the comparison information is. Moreover, the more elements $H_S^{ij^k}$ contains, the more hesitant the HFLTS is. So, the uncertainty of $H_S^{ij^k}$ should be measured by the comprehensive entropy defined by Eq. (22), from which the total uncertainty degree of HFLPR can be calculated. In the decision making processes, we usually expect that the uncertainty degree of the HFLPR is as small as possible to obtain results more precise. Thus, the bigger the uncertainty degree of $R^{(k)}$, the smaller the weight given to the corresponding expert e_k.

According to above analysis, the following Algorithm 2 is developed to assign the experts weights.

Algorithm 2. Step 1. Calculate the total comprehensive entropies $E_{c\beta}(R^{(k)})$ of $R^{(k)}$ by Eq. (22):

$$E_{c\beta}(R^{(k)}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E_{c\beta}\left(H_S^{ij^k}\right), \quad k = 1, 2, \ldots, m. \tag{33}$$

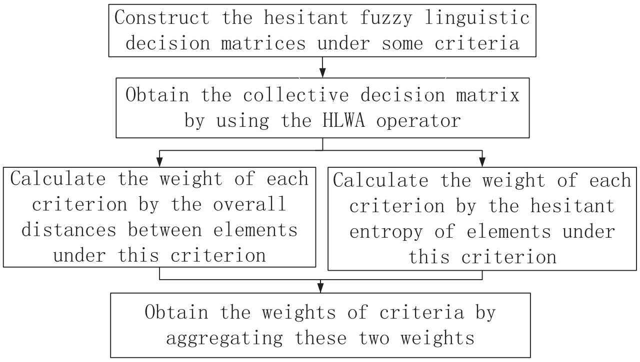

Fig. 2. Structure of Algorithm 2.

Since $H_S^{ji^k}$ is the negation of $H_S^{ij^k}$, and $H_S^{ii^k} = \{s_{\frac{g}{2}}\}$ for all $i, j = 1, 2, \ldots, n$, we have

$$E_{c\beta}(R^{(k)}) = \frac{1}{n} + \frac{2}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E_{c\beta}\left(H_S^{ij^k}\right), \quad k = 1, 2, \ldots, m. \tag{34}$$

Step 2. Calculate the weights w_k of the experts e_k:

$$w_k = \frac{1 - E_{c\beta}(R^{(k)})}{\sum_{i=1}^{m}(1 - E_{c\beta}(R^{(i)}))}, \quad k = 1, 2, \ldots, m. \tag{35}$$

The structure of Algorithm 2 is depicted in Fig. 2.

In the process of assigning the experts weights by Algorithm 2, both the hesitation and fuzziness of hesitant fuzzy linguistic information is considered. The value of β can be determined according to the expert's concern about the two types of uncertainty, fuzziness and hesitation.

5. Two illustrative examples

This section presents two examples to show the usefulness of the methods proposed to compute the criteria and experts weights. A sensitive analysis is also introduced in the example to compute the experts weights.

5.1. Example of assigning the objective criteria weights

An example of selecting movies adapted from Farhadinia [7,17], is used to illustrate the proposed Algorithm 1.

Let suppose a movies recommended system and a company provides ratings on five movies $A_i (i = 1, 2, 3, 4, 5)$ with respect to four criteria: story (C_1), acting (C_2), visuals (C_3) and direction (C_4) by using the linguistic term set $S = \{s_0$: terrible, s_1: very bad, s_2: bad, s_3: medium, s_4: well, s_5: very well, s_6: perfect$\}$.

To obtain more objective evaluations, the company asks a group of experts to assess the movies. Due to the lack of information and time pressure, the experts may hesitate between several linguistic terms, and they prefer to use more complex expressions to represent their assessments. In this situation, HFLTSs can be used to describe their opinions. So we get the hesitant fuzzy linguistic decision matrix $R = (H_S^{ij})_{5\times 4}$ as follows:

$$R = \begin{pmatrix} \{s_1, s_2, s_3\} & \{s_3, s_4\} & \{s_3, s_4, s_5\} & \{s_4, s_5\} \\ \{s_3, s_4, s_5\} & \{s_4, s_5\} & \{s_3, s_4\} & \{s_3, s_4, s_5\} \\ \{s_5, s_6\} & \{s_4, s_5, s_6\} & \{s_4, s_5\} & \{s_5\} \\ \{s_3, s_4, s_5\} & \{s_2, s_3, s_4\} & \{s_4, s_5, s_6\} & \{s_4, s_5\} \\ \{s_2, s_3\} & \{s_3, s_4, s_5\} & \{s_3, s_4, s_5\} & \{s_3, s_4\} \end{pmatrix}.$$

The criteria weights are computed by using the Algorithm 1.

Step 1. The distance is computed by using the distance measure d_1 defined by Eq. (26), and based on Eq. (30), the criteria weights are obtained:

$$\omega^1 = (0.414, 0.244, 0.171, 0.171)^T.$$

Step 2. We adopt the entropy measure $E_{h\alpha}$ defined by Eq. (21) with $\alpha = 1$ to calculate the hesitant entropies of H_S^{ij}, and get the hesitant entropy matrix (E_{h1}^{ij}):

$$\begin{pmatrix} 0.500 & 0.375 & 0.500 & 0.375 \\ 0.500 & 0.375 & 0.375 & 0.500 \\ 0.375 & 0.500 & 0.375 & 0 \\ 0.500 & 0.500 & 0.500 & 0.375 \\ 0.375 & 0.500 & 0.500 & 0.375 \end{pmatrix}.$$

From Eq. (31), we can obtain the weighting vector ω^2 of criteria:

$$\omega^2 = (0.2366, 0.2366, 0.2366, 0.2903)^T.$$

Taking $\lambda = 0.5$ in Eq. (32), we obtain the weighting vector of criteria

$$\omega = (0.3253, 0.2403, 0.2038, 0.2307)^T.$$

By using the method introduced in [7] to determinate the criteria weights, $E_{F_1^\lambda}$ defined by Eq. (2) with $\lambda = 1$, the weighting vector ν of criteria is

$$\nu = (0.248, 0.235, 0.248, 0.269)^T,$$

which is different with the weighting vector ω^2.

Compared with Farhadinia's method, there are two different points in Algorithm 1.

(1) In the process of deriving the weights of criteria, Farhadinia considered the fuzziness of evaluation information, while Algorithm 1 only considers the hesitation of the evaluation information. This has been explained in Remark of Section 4.2.

(2) Farhadinia only considered the uncertainty information when deriving the weights of criteria, while Algorithm 1 considers both the derivations among evaluations and the uncertainty of each evaluation under a criterion.

The example is used only to illustrate how to derive the criteria weights by Algorithm 1. In order to rank the alternatives, we can apply a hesitant fuzzy linguistic decision-making method, such as the outranking approach and TOPSIS proposed in [28,31].

5.2. An example of assigning the experts weights

A college in a university wants to hire some teachers. Through the trail presentation and the display of academic achievements, four candidates X_1, X_2, X_3 and X_4 come into the last selection. Four experts e_1, e_2, e_3 and e_4 from the academic committee are invited to offer suggestions. Each expert provides the pair-wise comparison preferences using the linguistic term set $S = \{s_0$: extremely poor; s_1 : very poor; s_2 : poor; s_3 : slightly poor; s_4 : equal; s_5 : slightly good; s_6 : good; s_7 : very good; s_8 : extremely good$\}$.

According to the pair-wise comparison preferences of the four experts, we can construct the hesitant fuzzy linguistic preference relations $R^{(k)} = (H_S^{ij^{(k)}})_{4\times 4}$ $(k = 1, 2, 3)$, shown as follows:

$$R^{(1)} = \begin{pmatrix} \{s_4\} & \{s_4, s_5\} & \{s_5, s_6\} & \{s_6, s_7\} \\ \{s_3, s_4\} & \{s_4\} & \{s_4, s_5\} & \{s_5\} \\ \{s_2, s_3\} & \{s_3, s_4\} & \{s_4\} & \{s_4, s_5\} \\ \{s_1, s_2\} & \{s_3\} & \{s_3, s_4\} & \{s_4\} \end{pmatrix}, \quad R^{(2)} = \begin{pmatrix} \{s_4\} & \{s_5, s_6, s_7\} & \{s_2, s_3\} & \{s_6\} \\ \{s_1, s_2, s_3\} & \{s_4\} & \{s_2\} & \{s_4, s_5\} \\ \{s_5, s_6\} & \{s_6\} & \{s_4\} & \{s_6, s_7\} \\ \{s_2\} & \{s_3, s_4\} & \{s_1, s_2\} & \{s_4 \end{pmatrix},$$

$$R^{(3)} = \begin{pmatrix} \{s_4\} & \{s_5\} & \{s_6, s_7\} & \{s_6, s_7\} \\ \{s_3\} & \{s_4\} & \{s_4, s_5\} & \{s_5, s_6\} \\ \{s_1, s_2\} & \{s_3, s_4\} & \{s_4\} & \{s_5\} \\ \{s_1, s_2\} & \{s_2, s_3\} & \{s_3\} & \{s_4\} \end{pmatrix}, \quad R^{(4)} = \begin{pmatrix} \{s_4\} & \{s_4, s_5\} & \{s_3, s_4\} & \{s_1, s_2\} \\ \{s_3, s_4\} & \{s_4\} & \{s_1, s_2\} & \{s_0, s_1\} \\ \{s_4, s_5\} & \{s_6, s_7\} & \{s_4\} & \{s_3, s_4\} \\ \{s_6, s_7\} & \{s_7, s_8\} & \{s_4, s_5\} & \{s_4\} \end{pmatrix}.$$

In order to rank the four candidates, we should first aggregate the preference relations of the four experts to get a comprehensive preference relation. To do this, it is necessary to determinate the experts weights. Algorithm 2 is used to derive these weights. Here the comprehensive entropy $E_{c\beta}(\beta = 0.6)$ is calculated by combining E_f^2 and E_{h1}, or E_f^3 and E_{h1}, respectively.

For E_f^2 and E_{h1}, we use Eq. (22) to calculate the comprehensive entropies $E_{c0.6}$ of $H_S^{ji^k}$, then get the entropy matrices with respect to $R^{(i)} (i = 1, 2, 3, 4)$:

$$\begin{pmatrix} 1.0000 & 0.9804 & 0.8854 & 0.6884 \\ 0.9804 & 1.0000 & 0.9804 & 0.9443 \\ 0.8854 & 0.9804 & 1.0000 & 0.9804 \\ 0.6884 & 0.9443 & 0.9804 & 1.0000 \end{pmatrix}, \begin{pmatrix} 1.0000 & 0.7981 & 0.8854 & 0.7747 \\ 0.7981 & 1.0000 & 0.7747 & 0.9804 \\ 0.8854 & 0.7747 & 1.0000 & 0.6884 \\ 0.7747 & 0.9804 & 0.6884 & 1.0000 \end{pmatrix},$$

Table 4
The entropies and weights of the experts taking $\beta = 0.6$.

	(E_f^2, E_{h1})	(E_f^3, E_{h1})		(E_f^2, E_{h1})	(E_f^3, E_{h1})
$E_{c0.6}(R^{(1)})$	0.9324	0.9714	w_1	0.1370	0.1066
$E_{c0.6}(R^{(2)})$	0.8440	0.9232	w_2	0.3161	0.2862
$E_{c0.6}(R^{(3)})$	0.8949	0.9500	w_3	0.2130	0.1864
$E_{c0.6}(R^{(4)})$	0.8352	0.8871	w_4	0.3339	0.4208

Table 5
The entropies and weights of the experts taking different β.

β	(E_f^γ, E_{h1})	$E(R^{(1)})$	$E(R^{(2)})$	$E(R^{(3)})$	$E(R^{(4)})$	w_1	w_2	w_3	w_4
0	(E_f^2, E_{h1})	0.9215	0.8466	0.8744	0.8055	0.1422	0.2799	0.2275	0.3524
0.2	(E_f^2, E_{h1})	0.9255	0.8527	0.8807	0.8166	0.1420	0.2808	0.2275	0.3497
0.4	(E_f^2, E_{h1})	0.9292	0.8580	0.8863	0.8264	0.1416	0.2839	0.2274	0.3471
0.8	(E_f^2, E_{h1})	0.9353	0.8674	0.8960	0.8432	0.1412	0.2895	0.2270	0.3453
1	(E_f^2, E_{h1})	0.9380	0.8708	0.9002	0.8504	0.1407	0.2932	0.2265	0.3395
0	(E_f^3, E_{h1})	0.9665	0.9316	0.9407	0.8668	0.1138	0.2323	0.2014	0.4525
0.2	(E_f^3, E_{h1})	0.9683	0.9345	0.9439	0.8744	0.1137	0.2349	0.2011	0.4503
0.4	(E_f^3, E_{h1})	0.9700	0.9371	0.9468	0.8811	0.1132	0.2374	0.2008	0.4487
0.8	(E_f^3, E_{h1})	0.9728	0.9414	0.9514	0.8926	0.1125	0.2423	0.2010	0.4442
1	(E_f^3, E_{h1})	0.9733	0.9432	0.9538	0.8975	0.1150	0.2446	0.1990	0.4414

$$\begin{pmatrix} 1.0000 & 0.9443 & 0.6884 & 0.6884 \\ 0.9443 & 1.0000 & 0.9804 & 0.8854 \\ 0.6884 & 0.9804 & 1.0000 & 0.9443 \\ 0.6884 & 0.8854 & 1.0000 & 1.0000 \end{pmatrix}, \begin{pmatrix} 1.0000 & 0.9804 & 0.9804 & 0.6884 \\ 0.9804 & 1.0000 & 0.6884 & 0.3637 \\ 0.9804 & 0.6884 & 1.0000 & 0.9804 \\ 0.6884 & 0.3637 & 0.9804 & 1.0000 \end{pmatrix}.$$

Similarly, for E_f^3 and E_{h1}, we also use Eq. (22) to obtain the comprehensive entropies $E_{c0.6}$ of $H_S^{ji^k}$, then the entropy matrices with respect to $R^{(i)} (i = 1, 2, 3, 4)$ can be obtained as follows:

$$\begin{pmatrix} 1.0000 & 0.9969 & 0.9632 & 0.8271 \\ 0.9969 & 1.0000 & 0.9969 & 0.9903 \\ 0.9632 & 0.9969 & 1.0000 & 0.9969 \\ 0.8271 & 0.9903 & 0.9969 & 1.0000 \end{pmatrix}, \begin{pmatrix} 1.0000 & 0.9011 & 0.9632 & 0.9132 \\ 0.9011 & 1.0000 & 0.9132 & 0.9969 \\ 0.9632 & 0.9132 & 1.0000 & 0.8271 \\ 0.9132 & 0.9969 & 0.8271 & 1.0000 \end{pmatrix},$$

$$\begin{pmatrix} 1.0000 & 0.9903 & 0.8271 & 0.8271 \\ 0.9903 & 1.0000 & 0.9969 & 0.9632 \\ 0.8271 & 0.9969 & 1.0000 & 0.9903 \\ 0.8271 & 0.9632 & 1.0000 & 1.0000 \end{pmatrix}, \begin{pmatrix} 1.0000 & 0.9969 & 0.9969 & 0.8271 \\ 0.9969 & 1.0000 & 0.8271 & 0.4520 \\ 0.9969 & 0.8271 & 1.0000 & 0.9969 \\ 0.8271 & 0.4520 & 0.9969 & 1.0000 \end{pmatrix}.$$

By Eqs. (34) and (35), we get the entropies $E_{c\beta}(R^{(i)})$ of $R^{(i)}$ $(i = 1, 2, 3, 4)$ and the weighting vector $w = (w_1, w_2, w_3, w_4)$ of the experts, which are shown in Table 4.

From Table 4, we can see that the ranking of weights, $w_4 > w_2 > w_3 > w_1$, derived from two comprehensive entropies are the same.

Next, a sensitivity analysis is made based on the effect of changing the parameter β in the comprehensive entropies defined by Eq. (22) on the experts's weights. Taking different values for β in [0,1], the entropies of $R^{(i)}$ and the experts's weights can be calculated, which is showed in Table 5. The trend of the experts' weights regarding β by (E_f^2, E_{h1}) and (E_f^3, E_{h1}) are represented in Fig. 3. For clarification, $E_{c\beta}(R^{(i)})$ is represented by $E(R^{(i)})$ in the table.

The obtained results in Table 5 indicate that

(1) the comprehensive entropies of $R^{(i)}$ increase when the value of the parameter β in Eq. (22) increases.
(2) the weight of experts does not change much with the change of parameter β.

The reason for the phenomenon (2) is that the hesitant entropies of the absolute majority elements in $R^{(i)}$ are 0 or 0.3 and hesitant entropies are so small and the fuzzy entropies are relatively large, that no matter how the parameter β changes from 0 to 1, the influence of hesitant entropies on comprehensive entropies is relatively small.

We have only used the example to illustrate how to derive the experts weights by Algorithm 2. As to how to aggregate the preference relations and how to rank the candidates, we can recur to the aggregation operators and the decision making method proposed in [32].

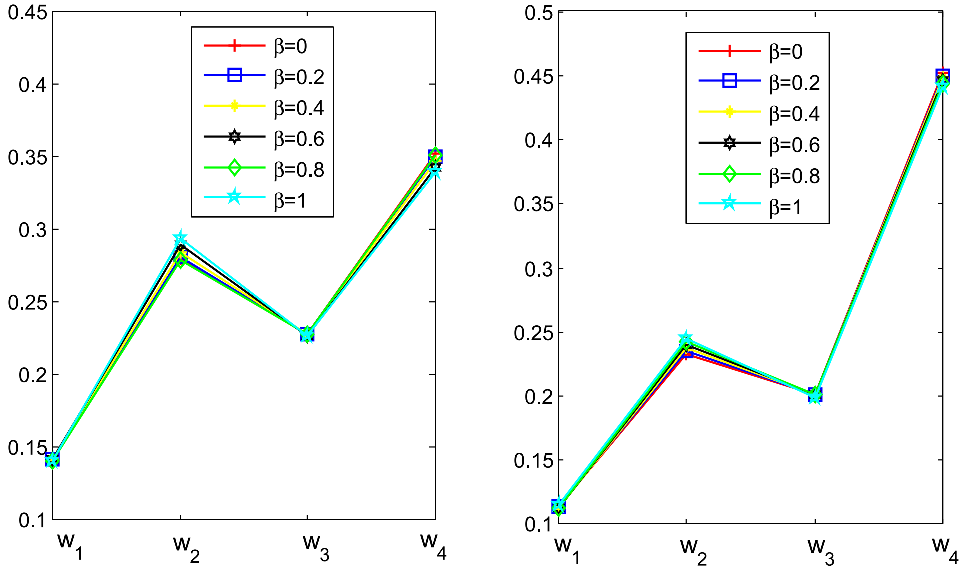

Fig. 3. Trend of the experts' weights regarding β by (E_f^2, E_{h1}) and (E_f^3, E_{h1}), respectively.

6. Conclusions

This paper points out the necessity of taking into account two types of uncertainty, fuzziness and hesitation, to define entropy measures for HFLTSs. Therefore, a new construction method of fuzzy entropy for HFLTSs has been proposed and a comparative analysis is performed to show the validity and effectiveness of the proposal. The results point out that the proposed entropy measures can distinguish between HFLTSs when the existing ones cannot. By using such entropy measures, two algorithms have been introduced to derive the criteria weights and the experts weights, respectively. We stressed that in different decision making settings, the different aspects of uncertainty should be considered. For example, if we want to measure the uncertainty of hesitant fuzzy linguistic evaluation information under some certain criteria, only the hesitant entropy should be adopted; if we wish to qualify the uncertainty of hesitant fuzzy linguistic information that representing the comparison preference of two items under a criterion, then both the fuzziness and the hesitancy of the decision making information should be considered, so, the comprehensive entropy should be used.

As future directions, we will further study fuzzy entropy measures for generalized HFLTS and other fuzzy extensions such as interval valued hesitant fuzzy sets [21] or intuitionistic fuzzy sets [11] and their application in different real world decision making problems.

Declaration of interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

Acknowledgments

The authors are very grateful to the editor and anonymous referees for their insightful and valuable suggestions that have led to an improved version of this paper. The work was partly supported by the National Natural Science Foundation of China (71371107, 71702087) and the Spanish Ministry of Economy and Competitiveness, Postdoctoral fellow Ramón y Cajal (RYC-2017-21978).

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