Short Papers

Uncertainty Measures of Extended Hesitant Fuzzy Linguistic Term Sets

Cuiping Wei, Rosa M. Rodríguez, and Luis Martínez

***Abstract*—A hesitant fuzzy linguistic term set (HFLTS) is defined as a subset of ordered consecutive linguistic terms, and it has been successfully applied to deal with experts hesitation in decision-making problems when experts have to provide their assessments. This concept has been recently extended to manage ordered consecutive and nonconsecutive linguistic terms, called extended HFLTS (EHFLTS), which is used in linguistic group decision-making problems to represent the group opinion without loss of information. This paper is focused on studying how to measure the uncertainty presented by the information of an EHFLTS and also of an HFLTS. To do so, a new comprehensive entropy measure for EHFLTSs, which considers two types of uncertainty, fuzziness and hesitation, is proposed. The construction methods of the two types of entropy are studied and a comprehensive entropy formula is defined. Finally, a comparative study is carried out to analyze the results obtained from the proposed entropy measures.**

***Index Terms*—Entropy, extended hesitant fuzzy linguistic term set (HFLTS), fuzziness, hesitation.**

I. Introduction

FUZZY sets theory has been widely and successfully applied in many different areas to manage imprecise and vague information [8]. However, it presents some limitations to deal with this type of information when several sources of vagueness appear simultaneously. In order to avoid such limitations, different extensions of fuzzy sets such as Atanassov intuitionistic fuzzy sets [1], type-2 fuzzy sets [4], fuzzy multisets [22], hesitant fuzzy sets [13], [17] have been proposed. These approaches are applied to problems defined in quantitative contexts, but the uncertainty is often because of the vagueness of meanings that are used by experts in qualitative contexts. In these situations, fuzzy linguistic approach [25] has provided very good results. Nevertheless, it is also limited mainly regarding the modeling of the linguistic information, because it is based on the use of single linguistic terms; sometimes due to the lack of knowledge and information, experts can hesitate among different linguistic terms to elicit their assessments and the use of only one linguistic term is not enough to represent their knowledge [11]. To deal with these hesitant situations, the concept of hesitant fuzzy linguistic term sets (HFLTS) has been introduced to model multiple linguistic terms and increase the flexibility to express the linguistic information by means of complex linguistic expressions [12].

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An HFLTS is defined as a finite set of consecutive linguistic terms, because it does not make sense to hesitate among different linguistic terms and not hesitate in their middle linguistic terms [13]. In group decision-making problems, multiple experts are involved and usually their assessments are aggregated to obtain a collective value that represents the group opinion. However, in order to keep as much information provided by the experts and avoid losing information in the aggregation process, an extension of HFLTS, called extended HFLTS (EHFLTS), has been introduced to deal even with nonconsecutive linguistic terms [18].

Entropy is a very important concept to measure the uncertainty associated with fuzzy sets. Zadeh introduced the concept of entropy of a fuzzy set in 1968 [24]. Since then, different researchers have studied the entropy measure for fuzzy sets [9], Atanassov intuitionistic fuzzy sets [2], hesitant fuzzy sets [3], etc. Another view of the concept of entropy measure was proposed in [16] for Atanassov intuitionistic fuzzy sets, which takes into account two types of uncertainty, fuzziness and hesitation, and it can be very useful for decision making dealing with Atanassov intuitionistic fuzzy sets.

Recently Farhadinia [6] presented several entropy measures for HFLTS based on distance and similarity measures, but such measures cannot be used for EHFLTS in spite of an HFLTS is a special case of an EHFLTS. In addition, such definitions only consider the fuzziness degree of an HFLTS, that is, the deviation of the linguistic terms contained in the HFLTS from the fuzziest element. From our view, not only the fuzziness should be considered to measure the entropy of an HFLTS, but also the hesitation of the HFLTS should be included in such a computation.

The aim of this paper is focused on how to improve the measurement of the uncertainty of an EHFLTS in a comprehensive manner so that it can capture both types of uncertainty, fuzziness and hesitation. The fuzziness of an EHFLTS can be described by the difference between the elements of the EHFLTS and the fuzziest value, that is, the middle term of the linguistic term set, and the hesitation can be represented by the deviation degree of the linguistic terms contained in the EHFLTS. Therefore, new entropy measures will be defined to represent the uncertainty related to an EHFLTS. Such entropy measures can be also applied to HFLTSs, which is a special case of EHFLTSs, and a comparison between different entropy measures for HFLTS will be developed. These entropy measures could be used for computing weights in multicriteria decision-making problems as other previous ones [3], [23].

The rest of this paper is structured as follows. Section II introduces some basic concepts about HFLTS, EHFLTS, and the fuzzy entropy measure. Section III studies the fuzziness and hesitation measures and defines a comprehensive entropy measure considering both types of uncertainty. Section IV presents a comparative analysis with other entropy measures for HFLTS. Section V concludes this paper.

II. Preliminaries

This section revises some concepts and basic operations about HFLTSs and their extension, EHFLTS, in which we focus on our proposal. It also revises the definition and meaning of fuzzy entropy.

A. HFLTSs and Basic Operations

The concept of HFLTS was introduced in [12] to model experts hesitation in qualitative contexts, because sometimes due to the lack of information and time pressure, experts feel doubtful in eliciting their preferences and the use of only one value is not enough to reflect their knowledge. In such situations, experts can express their preferences by using linguistic expressions based on HFLTSs.

Definition 1: [12] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set with an odd value of granularity and the middle term representing "approximately 0.5," an HFLTS H_S, is defined as an ordered finite subset of consecutive linguistic terms of S

$$H_S = \{s_i, s_{i+1}, \ldots, s_j\}\, such\, that\, s_k \in S,\ k \in \{i, \ldots, j\}. \tag{1}$$

Example 1: Let S be a linguistic term set $S = \{nothing, very\,bad, bad, medium, good, very\,good, perfect\}$ and ϑ a linguistic variable, then an HFLTS is

$$H_S(\vartheta) = \{good, very\,good, perfect\}.$$

Definition 2: Let $H_S = \{s_i, s_{i+1}, \ldots, s_j\}$ be an HFLTS and its cardinality is defined as the number of linguistic terms contained in the HFLTS.

Some operations such as the union, intersection, and negation were presented in [12].

Definition 3: [12] The union between two HFLTSs H_S^1 and H_S^2 is defined as

$$H_S^1 \cup H_S^2 = \{s_i | s_i \in H_S^1\, or\, s_i \in H_S^2\}. \tag{2}$$

Definition 4: [12] The intersection of two HFLTSs, H_S^1 and H_S^2 is

$$H_S^1 \cap H_S^2 = \{s_i | s_i \in H_S^1\, and\, s_i \in H_S^2\}. \tag{3}$$

Definition 5: The negation operator of an HFLTS, $H_S = \{s_i, s_{i+1}, \ldots, s_j\}$, is defined as follows:

$$\overline{H_S} = \{s_k\} \tag{4}$$

where $k = \{g - j, g - (j - 1), g - (j - 2), \ldots, g - i\}$ and $g + 1$ is the granularity of the linguistic term set S.

Despite this concept being quite novel, it has attracted the attention of many researchers, and different aggregation operators, distance measures, and decision-making methods have been presented in the literature [5]–[7], [19], [20].

B. Extended HFLTSs and Basic Operations

Recently, the concept of HFLTS has been generalized to deal with nonconsecutive linguistic terms [18]. An EHFLTS is built by means of the union of several HFLTSs.

Definition 6: [18] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set with an odd value of granularity and the middle term representing "approximately 0.5," an EHFLTS EH_S, is an ordered subset of linguistic terms of S, that is

$$EH_S = \{s_i | s_i \in S\}. \tag{5}$$

Example 2: Let $S = \{nothing, very\,bad, bad, medium, good, very\,good, perfect\}$ be a linguistic term set, and ϑ be a linguistic variable, then an EHFLTS is

$$EH_S(\vartheta) = \{bad, good, very\,good\}.$$

It is necessary to highlight that the concept of EHFLTS is not used directly to elicit linguistic information, but experts express their preferences by using HFLTSs and instead of carrying out an aggregation process to obtain the collective opinion, the group preferences are fused by means of the union operation of HFLTSs obtaining an EHFLTS.

Remark 1: An HFLTS is a special case of an EHFLTS in which the linguistic terms are consecutive.

Wang [18] introduced some basic operations for EHFLTSs.

Definition 7: [18] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and let EH_S^1 and EH_S^2 be two EHFLTSs, then the union between two EHFLTSs is defined as

$$EH_S^1 \cup EH_S^2 = \{s_i | s_i \in EH_S^1\, or\, s_i \in EH_S^2\}. \tag{6}$$

Definition 8: [18] The intersection of two EHFLTSs EH_S^1 and EH_S^2 is

$$EH_S^1 \cap EH_S^2 = \{s_i | s_i \in H_S^1\, and\, s_i \in H_S^2\}. \tag{7}$$

The result of both operations is another EHFLTS.

Definition 9: The negation operator of an EHFLTS EH_S is defined as follows:

$$\overline{EH_S} = \{s_{g-i} | s_i \in EH_S\}. \tag{8}$$

Some aggregation operators and distance measures for EHFLTS have been defined [18], [21].

C. Fuzzy Entropy Measure

In 1968, Zadeh introduced for the first time the concept of entropy of a fuzzy set, which may be interpreted as the uncertainty associated with a fuzzy event [24]. In 1972, De Luca and Termini proposed an axiomatic definition of the entropy for fuzzy sets [9] by using Shannon's entropy [14].

Let φ be the class of all the maps from I to L, where I is a finite set and L is a lattice. For each fuzzy set $f \in \varphi(I)$, a measure of the degree of its fuzziness is introduced, denoted by $d(f)$, which satisfies at least the following properties [9].

1) $d(f) = 0$, if only if f takes on I the values 0 or 1.
2) $d(f) = 1$, if only if f takes the value 0.5.
3) $d(f) \geq d(f^*)$ where f^* being any "sharpened" version of f, that is, any fuzzy set such that $f^*(x) \geq f(x)$ if $f(x) \geq 1/2$ and $f^*(x) \leq f(x)$ if $f(x) \leq 1/2$.

Definition 10: [9] Let $E(f)$ be a function on $\varphi(I)$, similar to the Shannon entropy, whose range is the set of nonnegative real numbers and defined as follows:

$$E(f) = -K \sum_{i=1}^{N} f(x_i) ln f(x_i) \tag{9}$$

N being the number of elements of I and K a positive constant.

Definition 11: [9] The fuzziness measure or "entropy" of a fuzzy set f is defined by

$$d(f) = E(f) + E(\overline{f}) \tag{10}$$

where $\overline{f} = 1 - f(x)$ satisfies the involution law and De Morgan laws (see [9] for further details).

The entropy introduced by De Luca and Termini was interpreted as "quantity of information" and it has been widely used in the fuzzy sets theory. Therefore, some proposals about entropy measures for fuzzy sets and their extensions are based on this definition [15]. Later on, a new view of entropy measure for Atanassov intuitionistic fuzzy sets was introduced in [16]. This new entropy measure not only considered the fuzziness, but also the hesitancy. In a similar way, in [26], an entropy measure for hesitant fuzzy sets, taking into account the fuzziness and nonspecificity, was presented.

III. New Comprehensive Entropy Measure for EHFLTSs

This section proposes a new framework to measure the uncertainty of an EHFLTS taking into account the fuzziness and hesitation. First, an axiomatic definition of entropy measure for EHFLTS is introduced. Second, several entropy measures for linguistic terms are presented and some construction methods of fuzzy and hesitant entropies are proposed for EHFLTS. Finally, a comprehensive entropy measure for EHFLTSs is introduced by combining the fuzzy and hesitant entropies.

A. Axiomatic Definition of Entropy for EHFLTS

The deviation function of an EHFLTS EH_S reflects the deviation degree (hesitancy) among the linguistic terms included in the EHFLTS. It is defined as follows.

Definition 12: Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $EH_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an EHFLTS on S. The deviation function of an EHFLTS EH_S is defined as follows:

$$\eta(EH_S) = \frac{2}{l(l-1)} \sum_{i=1}^{l-1} \sum_{j=i+1}^{l} \left(I(s_{\alpha_j}) - I(s_{\alpha_i}) \right) \qquad (11)$$

l being the cardinality of the EHFLTS EH_S and $I(s_{\alpha_k})$ the index of the linguistic term s_{α_k}.

Remark 2: If the EHFLTS consists of only one linguistic term $EH_S = \{s_{\alpha_1}\}$, then the deviation function $\eta(EH_S) = 0$.

Based on the deviation function $\eta(EH_S)$, an axiomatic definition of entropy for EHFLTSs is proposed.

Definition 13: Let $EH_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an EHFLTS on the linguistic term set S, and $\mathbb{EH}(S)$ be the set of all the EHFLTSs. Let E_f, E_h: $\mathbb{EH}(S) \to [0,1]$ be two mappings; if they satisfy the following axiomatic requirements, then E_f and E_h are the fuzzy and hesitant entropies of an EHFLTS, respectively.

F1) $E_f(EH_S) = 0$, if and only if $EH_S = \{s_0\}$ or $EH_S = \{s_g\}$ or $EH_S = \{s_0, s_g\}$.
F2) $E_f(EH_S) = 1$, if and only if $EH_S = \{s_{\frac{g}{2}}\}$.
F3) Let $EH_S^1 = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an EHFLTS, and EH_S^2 be another EHFLTS given by changing any element s_{α_i} $(i = 1, 2, \ldots, l)$ in EH_S^1 to $s_{\alpha_i'}$. If $|I(s_{\alpha_i}) - \frac{g}{2}| \geqslant |I(s_{\alpha_i'}) - \frac{g}{2}|$, then $E_f(EH_S^1) \leqslant E_f(EH_S^2)$.
F4) $E_f(EH_S) = E_f(\mathrm{Neg}(EH_S))$, where $\mathrm{Neg}(EH_S)$ is the negation operator of an EHFLTS EH_S.
H1) $E_h(EH_S) = 0$, if and only if $EH_S = \{s_{\alpha_1}\}$ (no hesitancy).
H2) $E_h(EH_S) = 1$, if and only if $EH_S = \{s_0, s_g\}$ (whole hesitancy).
H3) $E_h(EH_S^1) \leqslant E_h(EH_S^2)$, if $\eta(EH_S^1) \leqslant \eta(EH_S^2)$.
H4) $E_h(EH_S) = E_h(\mathrm{Neg}(EH_S))$.

B. Fuzzy Entropy Measures for Linguistic Terms

Following the axiomatic definition of entropy introduced previously, we propose an entropy measure for linguistic terms, which is the basis for generating some entropy formulas.

Definition 14: A real-valued function $E : S \to [0,1]$ is an entropy measure for a linguistic term $s_i \in S = \{s_0, \ldots, s_g\}$, if it satisfies the following axiomatic requirements.

E1) $E(s_i) = 0$, if and only if $s_i = s_0$ or $s_i = s_g$.
E2) $E(s_i) = 1$, if and only if $s_i = s_{\frac{g}{2}}$.
E3) $E(s_i) \leqslant E(s_j)$, if $s_i \leqslant s_j \leqslant s_{\frac{g}{2}}$ or $s_i \geqslant s_j \geqslant s_{\frac{g}{2}}$.
E4) $E(s_i) = E(\mathrm{Neg}(s_i))$.

The negation operator of a linguistic term is $\mathrm{Neg}(s_i) = s_{g-i}$.

Theorem 1: Let s_i be a linguistic term of a linguistic term set $S = \{s_0, \ldots, s_g\}$. $I(s_i)$ denotes the index of the linguistic term s_i

$$E(s_i) = f\left(\frac{I(s_i)}{g}\right) \qquad (12)$$

where the function $f : [0,1] \to [0,1]$ satisfies the following statements.

1) $f(1-x) = f(x)$, moreover, $f(0) = f(1) = 0$, $f(\frac{1}{2}) = 1$.
2) $f(x)$ is strictly monotone increasing when $x \in (0, 0.5]$ and strictly monotone decreasing when $x \in [0.5, 1)$.

Thus, $E(s_i)$ is an entropy measure for a linguistic term s_i.

Proof: It is sufficient to show the mapping $E(s_i)$, defined by (12), for satisfying the conditions (E1)–(E4) in Definition 14.

E1) $E(s) = f(\frac{I(s_i)}{g}) = 0$ if and only if $\frac{I(s_i)}{g} = 0$ or $\frac{I(s_i)}{g} = 1$ from the properties of the function f, that is, $s_i = s_0$ or $s_i = s_g$.
E2) $E(s) = f(\frac{I(s_i)}{g}) = 1$ if and only if $\frac{I(s_i)}{g} = \frac{1}{2}$, that is, $s_i = s_{\frac{g}{2}}$.
E3) If $s_i \leqslant s_j \leqslant s_{\frac{g}{2}}$, then $i \leqslant j$ and $f(\frac{i}{g}) \leqslant f(\frac{j}{g})$. Thus, $E(s_i) \leqslant E(s_j)$.
E4) Since $f(1-x) = f(x)$, we have $E(s_i) = f(\frac{i}{g}) = f(1 - \frac{i}{g}) = E(s_{g-i})$.

■

Theorem 2: Let $f : [0,1] \to R$ be a function and then the following statements hold:

1) $f''(x) < 0$ on $[0,1]$;
2) $f(x)$ is strictly monotone increasing on $(0, 0.5]$ and strictly monotone decreasing on $[0.5, 1)$;
3) $f(1-x) = f(x)$, moreover, $f(0) = f(1) = 0$, $f(0.5) = 1$;

which are satisfied if and only if there exists another function $g : [0,1] \to R$ such that $g''(x) < 0$, and

$$f(x) = \frac{h(x) - \min\limits_{0 \leqslant x \leqslant 1} h(x)}{\max\limits_{0 \leqslant x \leqslant 1} h(x) - \min\limits_{0 \leqslant x \leqslant 1} h(x)}$$

where $h(x) = g(x) + g(1-x)$.

Proof: The necessary condition can be obtained by letting $g(x) = f(x)$; therefore, we only prove the sufficiency. Let $g(x)$ be a real function on $[0,1]$ with $g''(x) < 0$, and $h(x) = g(x) + g(1-x)$ for $x \in [0,1]$. Then, $h(x) = h(1-x)$ and $h''(x) = g''(x) + g''(1-x) < 0$ for each $x \in [0,1]$. Since $g'(x)$ is decreasing on $[0,1]$ and $h'(x) = g'(x) - g'(1-x)$, we have $h'(x) > 0$ for $x \in (0, 0.5)$, and $h'(x) < 0$ for $x \in (0.5, 1]$, so $h(x)$ is strictly monotone increasing on $(0, 0.5]$ and strictly monotone decreasing on $[0.5, 1)$. Consequently, $h(0) = h(1) = \min\limits_{0 \leqslant x \leqslant 1} h(x)$ and $h(0.5) = \max\limits_{0 \leqslant x \leqslant 1} h(x)$. If $f(x) = \frac{h(x) - h(0)}{h(0.5) - h(0)}$ for $x \in [0,1]$, then $f(x)$ satisfies the three conditions mentioned. ■

From Theorems 1 and 2, we can construct different entropy formulas by using different convex functions g selected [10].

Definition 15: Let $g : [0,1] \to \infty$ be a function defined as $g(x) = ax - bx^2$ with $0 < b \le a$. Then, $g''(x) < 0$. Since $h(x) = 2bx(1-x) + a - b$, $\min\limits_{0 \leqslant x \leqslant 1} h(x) = a - b$, and $\max\limits_{0 \leqslant x \leqslant 1} h(x) = \frac{b}{2} + a - b$, we have $f(x) = 4x(1-x)$. From Theorems 1 and 2, the entropy function of a linguistic term $s_i \in S$ can be defined as follows:

$$E_1(s_i) = 4\frac{I(s_i)}{g}\left(1 - \frac{I(s_i)}{g}\right). \quad (13)$$

Definition 16: Let $g : [0,1] \to \infty$ be a function defined as $g(x) = -x\log x$. Then, $g''(x) < 0$. We have $h(x) = -x\log x - (1-x)\log(1-x)$, $\min_{0\leqslant x\leqslant 1} h(x) = 0$, $\max_{0\leqslant x\leqslant 1} h(x) = \log 2$, and $f(x) = \frac{-x\log x-(1-x)\log(1-x)}{\log 2}$. Therefore, the entropy function of a linguistic term $s_i \in S$ is defined as follows:

$$E_2(s_i) = \frac{1}{\log 2}\left[-\frac{I(s_i)}{g}\log\left(\frac{I(s_i)}{g}\right) - \left(1-\frac{I(s_i)}{g}\right)\log\left(1-\frac{I(s_i)}{g}\right)\right]. \quad (14)$$

Definition 17: Let $g : [0,1] \to \infty$ be a function defined as $g(x) = xe^{1-x}$. Then, $g''(x) < 0$. We have $h(x) = xe^{(1-x)} + (1-x)e^x$ and $f(x) = \frac{xe^{(1-x)}+(1-x)e^x-1}{e^{\frac{1}{2}}-1}$. Therefore, the entropy function of a linguistic term is as follows:

$$E_3(s_i) = \frac{\frac{I(s_i)}{g}e^{\left(1-\frac{I(s_i)}{g}\right)} + \left(1-\frac{I(s_i)}{g}\right)e^{\frac{I(s_i)}{g}} - 1}{e^{\frac{1}{2}}-1}. \quad (15)$$

Definition 18: Let $g : [0,1] \to \infty$ be a function defined as $g(x) = x^\delta (0 < \delta < 1)$. Then, $g''(x) < 0$. Since $h(x) = x^\delta + (1-x)^\delta$, $\min_{0\leqslant x\leqslant 1} h(x) = 1$, and $\max_{0\leqslant x\leqslant 1} h(x) = 2^{1-\delta}$, we have $f(x) = \frac{x^\delta+(1-x)^\delta-1}{2^{1-\delta}-1}$. So the entropy function of a linguistic term $s_i \in S$ is

$$E_4(s_i) = \frac{\left(\frac{I(s_i)}{g}\right)^\delta + \left(1-\frac{I(s_i)}{g}\right)^\delta - 1}{2^{1-\delta}-1}. \quad (16)$$

C. Construction of the Fuzzy Entropy Measure for an EHFLTS

In order to achieve our goal of improving the measurement of the uncertainty of an EHFLTS considering two types of uncertainty, fuzziness and hesitation, in this section, some fuzzy entropy measures for EHFLTSs are proposed based on the entropy formulas defined previously for linguistic terms $\{E_1, E_2, E_3, E_4\}$.

Theorem 3: Let $EH_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an EHFLTS. A general fuzzy entropy measure of an EHFLTS $E_f(EH_S)$ is defined by

$$E_f(EH_S) = \frac{1}{l}\sum_{i=1}^{l} E(s_{\alpha_i}). \quad (17)$$

Proof: Keeping in mind that the entropy measure for linguistic terms introduced by (12) satisfies the conditions (E1)–(E4), it is easy to prove that the mapping $E_f(EH_S)$ defined by (17) satisfies the conditions (F1)–(F4) in the Definition 13. ■

The fuzzy entropy measures for EHFLTSs are introduced as follows:

$$E_{f1}(EH_S) = \frac{1}{l}\sum_{i=1}^{l} 4\frac{I(s_{\alpha_i})}{g}\left(1-\frac{I(s_{\alpha_i})}{g}\right) \quad (18)$$

$$E_{f2}(EH_S) = \frac{1}{l}\sum_{i=1}^{l}\frac{1}{\log 2}\left[-\frac{I(s_{\alpha_i})}{g}\log\left(\frac{I(s_{\alpha_i})}{g}\right) - \left(1-\frac{I(s_{\alpha_i})}{g}\right)\log\left(1-\frac{I(s_{\alpha_i})}{g}\right)\right] \quad (19)$$

$$E_{f3}(EH_S) = \frac{1}{l}\sum_{i=1}^{l}\frac{\frac{I(s_{\alpha_i})}{g}e^{\left(1-\frac{I(s_{\alpha_i})}{g}\right)} + \left(1-\frac{I(s_{\alpha_i})}{g}\right)e^{\frac{I(s_{\alpha_i})}{g}} - 1}{e^{\frac{1}{2}}-1} \quad (20)$$

$$E_{f4}(EH_S) = \frac{1}{l}\sum_{i=1}^{l}\frac{\left(\frac{I(s_{\alpha_i})}{g}\right)^\delta + \left(1-\frac{I(s_{\alpha_i})}{g}\right)^\delta - 1}{2^{1-\delta}-1}. \quad (21)$$

D. Construction of the Hesitant Entropy Measure for an EHFLTS

The hesitant entropy measure of an EHFLTS is based on the deviation function $\eta(EH_S)$ introduced in Definition 12, which reflects the deviation degree of the linguistic terms present in the EHFLTS.

Theorem 4: Let $EH_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an EHFLTS on the linguistic term set $S = \{s_0, \ldots, s_g\}$, and let $f : [0,1] \to [0,1]$ be a basic unit-interval monotonic (BUM) function that satisfies the following two conditions.

1) $f(0) = f(0)$, $f(1) = f(1)$.
2) $f(x)$ is strictly monotone increasing on $(0,1]$.

Then, a hesitant entropy measure for an EHFLTS EH_S is defined as follows:

$$E_h(H_S) = f\left(\frac{1}{g}\eta(H_S)\right). \quad (22)$$

Proof: It is sufficient to prove the mapping E_h defined by (22), which satisfies the requirements (H1)–(H4) in Definition 13.

H1) $E_h(EH_S) = f(\frac{1}{g}\eta(EH_S)) = 0$ if and only if $\eta(H_S) = 0$, and $\eta(H_S) = 0$ if and only if the EHFLTS includes only one linguistic term, that is, $EH_S = \{s_i\}(i = 0, 1, \ldots, g)$.

H2) $E_h(EH_S) = f(\frac{1}{g}\eta(EH_S)) = 1$ if and only if $\eta(EH_S) = \frac{2}{l(l-1)}\sum_{i=1}^{l-1}\sum_{j=i+1}^{l}(I(s_{\alpha_j}) - I(s_{\alpha_i})) = g$, if and only if $EH_S = \{s_0, s_g\}$.

H3) Since $f(x)$ is strictly monotone increasing on $(0,1]$, we can obtain the conclusion that if $\eta(EH_S^1) \leqslant \eta(EH_S^2)$, then $E_h(EH_S^1) \leqslant E_h(EH_S^2)$.

H4) If $\eta(EH_S) = \eta(\text{Neg}(EH_S))$, then $E_h(EH_S) = E_h(\text{Neg}(EH_S))$. ■

It is necessary to highlight that if the function $f(x)$ in $E_h(EH_S)$ defined by (22) is changed, a family of entropy measures for EHFLTSs is obtained. For example, suppose that $f(x) = x^\gamma$ with $\gamma > 0$, the hesitant entropy formula is

$$E_{h\gamma}(EH_S) = \left(\frac{1}{g}\eta(EH_S)\right)^\gamma. \quad (23)$$

E. Comprehensive Entropy Measure for an EHFLTS

Once, we have studied the fuzzy and hesitant measures to deal with the uncertainty of an EHFLTS, both measures are combined to obtain an entropy measure that is able to reflect both types of uncertainty for an EHFLTS.

Definition 19: Let $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an EHFLTS on a linguistic term set, $S = \{s_0, s_1, \ldots, s_g\}$. Let $E_c : \mathbb{EH}(S) \to [0,1]$ be a mapping. Then, $E_c(EH_S)$ is a comprehensive entropy of an EHFLTS EH_S, if it satisfies the following statements.

E1) $E_c(EH_S) = 0$, if and only if $EH_S = \{s_0\}$, $EH_S = \{s_g\}$.

E2) $E_c(EH_S) = 1$, if and only if $EH_S = \{s_{\frac{g}{2}}\}$.

E3) Let $EH_S^1 = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an EHFLTS, and EH_S^2 be another EHFLTS given by changing any element $s_{\alpha_i}(i = 1, 2, \ldots, l)$ in EH_S^1 to $s_{\alpha_i'}$. If $|I(s_{\alpha_i}) - \frac{g}{2}| \geqslant |I(s_{\alpha_i'}) - \frac{g}{2}|$ and $\eta(EH_S^1) \leqslant \eta(EH_S^2)$, then $E_c(EH_S^1) \leqslant E_c(EH_S^2)$.

E4) $E_c(EH_S) = E_c(\text{Neg}(EH_S))$.

Theorem 5: If a real-valued function $E_c : \mathbb{EH} \to [0,1]$ is defined by $E_c(EH_S) = f(E_f(EH_S), E_h(EH_S))$, where the function $f : [0,1] \times [0,1] \to [0,1]$ satisfies the following conditions:

1) $f(0,0) = 0, f(1,0) = 1$;
2) $f(x,y)$ is strictly monotone increasing with respect to x and y, respectively;

then the mapping E_c is a comprehensive entropy measure of the EHFLTS EH_S.

Proof: It is sufficient to prove that the function E_c satisfies the requirements (E1)–(E4) in Definition 19.

E1) $E_c(EH_S) = f(E_f(EH_S), E_h(EH_S)) = 0$, if and only if $E_f(EH_S) = 0, E_h(EH_S) = 0$, if and only if $EH_S = \{s_0\}$, $EH_S = \{s_g\}$.

E2) $E_c(EH_S) = f(E_f(EH_S), E_h(EH_S)) = 1$, if and only if $E_f(EH_S) = 1, E_h(EH_S) = 0$, if and only if $EH_S = \{s_{\frac{g}{2}}\}$.

E3) Let $EH_S^1 = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an EHFLTS and EH_S^2 be another EHFLTS given by changing any element $s_{\alpha_i}(i = 1,2,\ldots,l)$ in EH_S^1 to $s_{\alpha_i'}$, if $|I(s_{\alpha_i}) - \frac{g}{2}| \geqslant |I(s_{\alpha_i'}) - \frac{g}{2}|$ and $\eta(EH_S^1) \leqslant \eta(EH_S^2)$, then $E_f(EH_S^1) \leqslant E_f(EH_S^2)$ and $E_h(EH_S^1) \leqslant E_h(EH_S^2)$. Since $f(x,y)$ is strictly monotone increasing with respect to x and y, then $E_c(EH_S^1) \leqslant E_c(EH_S^2)$.

E4) From $E_f(EH_S) = E_f(\text{Neg}(EH_S))$ and $E_h(EH_S) = E_h(\text{Neg}(EH_S))$, then $E_c(EH_S) = E_c(\text{Neg}(EH_S))$. ∎

Different comprehensive entropy measures for EHFLTSs can be defined using several special functions $f(x)$. Let us see the following example.

Example 3: Let $f : [0,1] \times [0,1] \to [0,1]$ be defined as $f(x,y) = \frac{x+\beta y}{1+\beta y}$. Then, $f(x,y)$ satisfies conditions 1 and 2 in Theorem 5. Therefore, the comprehensive entropy measure of an EHFLTS EH_S is given by

$$E_{c\beta}(EH_S) = \frac{E_f + \beta E_h}{1 + \beta E_h} \tag{24}$$

with $\beta \in [0,1]$.

The parameter β can be fixed according to the importance of the hesitation of an EHFLTS.

IV. Comparison Study About Entropy Measures

This section makes a comparative analysis among the proposed entropy measures for EHFLTSs and some entropy measures introduced for HFLTSs, which is a special case of EHFLTSs. To do so, two entropy measures for HFLTSs are revised and an example is introduced to compute the entropy with the proposed entropy measures and the revised ones for HFLTS. Afterward, an analysis of the results obtained is carried out.

A. Entropy Measures for HFLTSs

As far as we know, there is no any study of the entropy measure for an EHFLTS and there is only one paper introduced by Farhadinia [6] that studies the relationship between the entropy measure and the similarity measure for HFLTS and proposes a variety of entropy measures for HFLTSs based on the distance and similarity measures. Therefore, the entropy measures based on the generalized distance and generalized similarity proposed by Farhadinia are revised, the remaining ones are special cases of these two definitions and can be found in [6].

For convenience to carry out the comparison in a proper way, the entropy measures defined by Farhadinia [6] on the linguistic term set $S' = \{s_{-\tau}, \ldots, s_{-1}, s_0, s_1, \cdots, s_\tau\}$ are adapted to use the same linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$ used in the proposed entropy measures.

Definition 20: Let $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an HFLTS defined on the linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$, the entropy measure based on the generalized distance is defined by

$$E_{F_1^\lambda} = 1 - 2\left(\frac{1}{l}\sum_{i=1}^{l}\left(\frac{|I(s_{\alpha_i}) - g/2|}{g}\right)^\lambda\right)^{\frac{1}{\lambda}} \tag{25}$$

l being the cardinality of the HFLTS, H_S, and $\lambda > 0$.

Definition 21: Let $H_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an HFLTS defined on the linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$, the entropy measure based on the generalized similarity is defined as follows:

$$E_{F_2^\lambda} = 1 - \left(\frac{1}{l}\sum_{i=1}^{l}\left(\frac{|I(s_{\alpha_i}) - g/2|}{g/2}\right)^\lambda\right)^{\frac{2}{\lambda}} \tag{26}$$

l being the cardinality of the HFLTS H_S and $\lambda > 0$.

B. Computations and Comparisons

Some examples of HFLTSs and EHFLTSs are introduced and their entropies are calculated by the proposed entropy measures E_{f1}, E_{f2}, E_{f3}, E_{f4} (with $\delta = 0.5$), $E_{h\gamma}$ (with $\gamma = 1$), and $E_{c\beta}$ (with $\beta = 0.4$), respectively. The entropy measures for HFLTSs $E_{F_1^\lambda}$ and $E_{F_2^\lambda}$ (with $\lambda = 1$), are also computed. The results are shown in Table I.

Example 4: Let $S = \{s_0 : nothing, s_1 : very\,low, s_2 : low, s_3 : slightly\,low, s_4 : medium, s_5 : slightly\,high, s_6 : high, s_7 : very\,high, s_8 : perfect\}$ be a linguistic term set, and let

$$H_S^1 = \{s_0\},\ H_S^2 = \{s_1\},\ H_S^3 = \{s_0, s_1, s_2\},\ H_S^4 = \{s_3\}$$
$$H_S^5 = \{s_2, s_3, s_4\},\ H_S^6 = \{s_4\},\ EH_S^7 = \{s_0, s_2, s_3\}$$
$$EH_S^8 = \{s_0, s_2, s_4\},\ H_S^9 = \{s_6, s_7, s_8\}$$

be nine EHFLTSs in which seven are HFLTSs.

C. Analysis of Results

Before introducing the analysis of the results obtained in Table I, the following definition is revised because it is used in the analysis.

Definition 22: [18] Let $EH_S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$ be an EHFLTS on the linguistic term set $S = \{s_0, \ldots, s_g\}$, the average of an EHFLTS is given by

$$\theta(EH_S) = s_\alpha \ such\ that\ \alpha = \frac{1}{l}\sum_{i=1}^{l} I(s_{\alpha_i}). \tag{27}$$

It is easy to see in Table I that the results shown in bold type reflect some counter-intuition examples regarding the entropy measures $E_{F_1^\lambda}$ and $E_{F_2^\lambda}$. For instance, if we consider H_S^2 and H_S^3, we can see that $\theta(H_S^2) = \theta(H_S^3) = s_1$, but the result of the deviation function shows that $\eta(H_S^2) < \eta(H_S^3)$. Thus, the uncertainty of H_S^2 and H_S^3 should be different. However, according to the entropy measures $E_{F_1^1}$ and $E_{F_2^1}$, we can see that $E_{F_1^1}(H_S^2) = E_{F_1^1}(H_S^3)$ and $E_{F_2^1}(H_S^2) = E_{F_2^1}(H_S^3)$, which is not consistent with our intuition.

The proposed entropy measures $E_{c\beta}$ are effective because they reflect the hesitation and fuzziness, and therefore the results obtained are different; thus, it is possible to distinguish between H_S^2 and H_S^3. Similarly for H_S^4 and H_S^5.

Farhadinia entropy measures cannot be applied to the EHFLTS, EH_S^7 and EH_S^8; however, the proposed entropy measures offer more flexibility and they can be applied to both HFLTS and EHFLTS.

TABLE I
THE ENTROPIES OF $H_S^i (i = 1, 2, \ldots, 9)$ BY DIFFERENT ENTROPY MEASURES

	H_S^1	H_S^2	H_S^3	H_S^4	H_S^5	H_S^6	EH_S^7	EH_S^8	H_S^9
$\eta(H_S)$	0	0	1.333	0	1.333	0	2	2.667	1.336
$E_{f1}(H_S^i)$	0	0.438	0.396	0.938	0.896	1	0.563	0.583	0.396
$E_{f2}(H_S^i)$	0	0.544	0.452	0.954	0.922	1	0.588	0.604	0.452
$E_{f3}(H_S^i)$	0	0.449	0.403	0.940	0.900	1	0.566	0.586	0.403
$E_{f4}(H_S^i)$	0	0.698	0.527	0.973	0.952	1	0.619	0.628	0.527
$E_{h1}(H_S^i)$	0	0	0.167	0	0.167	0	0.25	0.33	0.167
$E_{c0.4} = f(E_{f1}, E_{h1})$	0	0.438	0.434	0.938	0.903	1	0.603	0.632	0.434
$E_{c0.4} = f(E_{f2}, E_{h1})$	0	0.544	0.486	0.954	0.927	1	0.625	0.651	0.486
$E_{c0.4} = f(E_{f3}, E_{h1})$	0	0.449	0.440	0.940	0.906	1	0.606	0.635	0.440
$E_{c0.4} = f(E_{f4}, E_{h1})$	0	0.698	0.557	0.973	0.955	1	0.654	0.672	0.557
$E_{F_1^1}(H_S^i)$	0	**0.250**	**0.250**	**0.750**	**0.750**	1	–	–	0.25
$E_{F_2^1}(H_S^i)$	0	**0.438**	**0.438**	**0.938**	**0.938**	1	–	–	0.438

It is remarkable to mention that the EHFLTS EH_S^7 and EH_S^8 satisfy the condition $E3$ of the Definition 19, because $|I(s_3) - g/2| \geq |I(s_4) - g/2|$ and $\eta(H_S^7) \leq \eta(H_S^8)$ and then $E_{c0.4}(EH_S^7) \leq E_{c0.4}(EH_S^8)$. It is easy to see that $\mathrm{Neg}(H_S^3) = H_S^9$, and the results obtained from $E_{c0.4}(H_S^3) = E_{c0.4}(H_S^9)$, therefore, it satisfies the condition $E4$ of Definition 19.

V. CONCLUSION

This paper shows the necessity of considering two types of uncertainty associated with EHFLTS for measuring their entropy in an intuitive way. The first kind is associated with the fuzziness in the set and the second one is the hesitation represented by the deviation degree of the linguistic terms contained in the EHFLTS. Consequently, we have proposed the construction methods of the hesitant and fuzzy entropy measures and introduced a comprehensive entropy measure that considers both types of uncertainty. These entropy measures can be applied to EHFLTS and HFLTS, which is a special case of an EHFLTS. A comparison with previous entropy measures defined for HFLTS has been carried out to show the validity and effectiveness of our proposal. The proposed entropy measures can be directly applied to compute the weights in multicriteria decision-making problems dealing with HFLTS and EHFLTS in their definition contexts.

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