Stochastic Bernstein polynomials: uniform convergence in probability with rates

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Abstract We introduce stochastic variants of the classical Bernstein polynomials associated to a continuous function f, built up from a general triangular array of random variables. We discuss the uniform convergence in probability of the ap-proximation process that they represent, providing at the same time rates of con-vergence. In the particular case in which the triangular array of random variables consists of the uniform order statistics, we give a positive answer to a conjectured raised in [15] about an exponential rate of convergence in probability.

Keywords stochastic Bernstein polynomials · uniform convergence in probability · rates of convergence · confidence band · Bernstein-Durrmeyer polynomials

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1 Introduction

Throughout this paper, $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, and $x \in [0, 1]$. We will always consider functions f in the space $C[0, 1]$, $\|f\|_\infty$ refers to their supremum norm, and $C^k[0, 1]$ denotes the subspace of all k times continuously differentiable functions.

The classical Bernstein polynomials defined by

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \tag{1}$$

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are the paradigmatic example of positive linear operators. Rates of uniform convergence for these operators are characterized by

$$K_1\omega_\varphi^2\left(f;\frac{1}{\sqrt{n}}\right) \le \|B_n(f;x)-f(x)\|_\infty \le K_2\omega_\varphi^2\left(f;\frac{1}{\sqrt{n}}\right), \tag{2}$$

for some positive absolute constants K_1 and K_2, where $\omega_\varphi^2(f;\cdot)$ stands for the Ditzian-Totic second modulus of continuity of f with weight function $\varphi(x) = \sqrt{x(1-x)}$ (see, for instance, Ditzian and Ivanov [6] or Totik [12]). Păltănea [9] gave $K_2 = 2.5$, whereas no specific value for K_1 has been provided, yet.

On the other hand, Sikkema [10] showed that

$$\|B_n(f;x)-f(x)\|_\infty \le c\omega\left(f;\frac{1}{\sqrt{n}}\right), \quad c = \frac{4306+837\sqrt{6}}{5832} = 1.089887\dots, \tag{3}$$

where c is the best possible constant and $\omega(f;\cdot)$ is the usual first modulus of continuity of f defined by

$$\omega(f;\delta) = \sup\{|f(x)-f(y)| : \ x,y\in[0,1], \ |x-y|\le\delta\}, \quad 0\le\delta\le 1.$$

In the definition of the Bernstein polynomials, it is implicitly assumed that one is able to evaluate the function f at the set of equally spaced nodes k/n, $k = 0,1,\dots,n$. However, in real problems, data at equally spaced sites are often unavailable and sometimes contaminated by random errors due to a variety of factors. These are the motivations behind the definition of the stochastic Bernstein polynomials recently introduced by Wu *et al.* [14], further developed in [11,15], and extended to a multivariate setting in [5].

More precisely, let $\mathbb{Y} = (Y_{n,k}, \ n\in\mathbb{N}, \ k=0,1,\dots,n)$ be a triangular array of random variables such that

$$0 \le Y_{n,0} \le Y_{n,1} \le \dots \le Y_{n,n} \le 1, \quad n\in\mathbb{N}. \tag{4}$$

Wu et all. [14] define the nth stochastic Bernstein polynomial as

$$B_n(f,\mathbb{Y};x) = \sum_{k=0}^n f(Y_{n,k})p_{n,k}(x). \tag{5}$$

Observe that formula (5) defines a continuous random function on $[0,1]$, not necessarily interpolating f at the endpoints, or a continuous deterministic function in the case in which $Y_{n,k}$ are deterministic points.

Recall that a sequence of random variables $(X_n)_{n\ge1}$ converges to 0 in probability, denoted by $X_n \xrightarrow{(P)} 0$, if

$$\lim_{n\to\infty} P(|X_n|>\epsilon) = 0, \quad \epsilon>0.$$

In the stochastic setting outlined above, the analogous results to (2) and (3) consist of analyzing the uniform convergence in probability

$$P\left(\|B_n(f,\mathbb{Y};x)-f(x)\|_\infty > \epsilon\right), \quad \epsilon>0,$$

trying at the same time to give rates of convergence. In other words, to construct a confidence band $[f(x)-\epsilon, f(x)+\epsilon]$, $x\in[0,1]$, for the random function $B_n(f,\mathbb{Y};x)$.

An important special case of definition (5) is the following. For each $n \in \mathbb{N}$, let $(V_j)_{j=1}^{n+1}$ be a finite sequence of independent identically distributed random variables having the uniform distribution on $[0,1]$. Let $V_{n+1:1} \leq \ldots \leq V_{n+1:n+1}$ be the order statistics obtained by arranging $(V_j)_{j=1}^{n+1}$ in increasing order of magnitude. Define

$$Y_{n,k} = V_{n+1:k+1}, \quad n \in \mathbb{N}, \quad k = 0, 1, \ldots, n. \tag{6}$$

By construction, the triangular array defined in (6) satisfies condition (4). In addition, the random variable $Y_{n,k} = V_{n+1:k+1}$ has the beta probability density (see, for instance, Arnold et al. [4, Chap. 2])

$$\rho_k(\theta) = (n+1)p_{n,k}(\theta), \quad \theta \in [0,1], \quad n \in \mathbb{N}, \quad k = 0, 1, \ldots, n, \tag{7}$$

where $p_{n,k}(\theta)$ is defined in (1). For the triangular array in (6), Wu and Zhou [15] have obtained the estimate

$$P\big(\|B_n(f, \mathbb{Y}; x) - f(x)\|_\infty > \epsilon\big) \leq C(r)\frac{n\omega^{2r}(f; 1/\sqrt{n})}{\epsilon^{2r}}, \quad \epsilon > 0, \tag{8}$$

for any $r \in \mathbb{N}$, whenever $\omega(f; 1/\sqrt{n}) < \epsilon/4.2$, where $C(r)$ is a constant depending only on r. The case $r = 3$, with $C(r) = C(3) = 15/64$ had already been obtained by Wu et al in [14]. Besides, Sun and Wu [11] have proved the estimate

$$P\big(\|B_n(f, \mathbb{Y}; x) - f(x)\|_\infty > \epsilon\big) \leq 40\frac{\omega^2(f; 1/\sqrt{n})}{\epsilon^2}, \quad \epsilon > 0,$$

whenever $\omega(f; 1/\sqrt{n}) < \epsilon/7.2$.

Note that estimate (8) guarantees uniform convergence in probability when f belongs to any Hölder class. On the other hand, Wu and Zhou [15] conjectured that for large enough n

$$P\left(\|B_n(f, \mathbb{Y}; x) - f(x)\|_\infty > \epsilon\right) \leq Cn \exp\left(-\frac{1}{4}\frac{\epsilon^2}{\omega^2(f; 1/\sqrt{n})}\right), \tag{9}$$

for some positive constant C.

In this paper, we show that, under the general setting given for $\mathbb{Y}$ in (4), and under the assumption that

$$M_n := \max_{0 \leq k \leq n}\left|Y_{n,k} - \frac{k}{n}\right| \xrightarrow{(P)} 0, \tag{10}$$

for any function $f \in C[0,1]$

$$\lim_{n\to\infty} P\big(\|B_n(f, \mathbb{Y}; x) - f(x)\|_\infty > \epsilon\big) = 0, \quad \epsilon > 0,$$

and we provide at the same time rates of convergence. The results are applied to the particular case given by the order statistics, and we give a positive answer to the conjecture written in (9).

The double inequality in (2) for the classical Bernstein polynomials implies that rates of convergence are improved for smooth functions f. This leads us to consider the following variant of the stochastic Bernstein polynomials defined in (5). Assume that $f \in C^2[0,1]$ and, what is more important, that we can observe the values of the first derivative f' at the random points $Y_{n,k}$. Under such circumstances, we define stochastic Bernstein-type polynomials as

$$B_n^\star(f,\mathbb{Y};x)=\sum_{k=0}^{n}\big(f(Y_{n,k})-f'(Y_{n,k})(Y_{n,k}-x)\big)p_{n,k}(x). \tag{11}$$

Such stochastic polynomials preserve affine functions, i.e., if $f(x)=ax+b$, for some real constants a and b, then

$$B_n^\star(f,\mathbb{Y};x)=B_n(f;x)=f(x).$$

As shown in Theorem 1 and Corollary 3 below, such stochastic operators approximate functions $f\in C^2[0,1]$ at better rates of convergence than those defined in (5).

2 Uniform convergence in probability

Let $I=[a,b]$ and $J=[A,B]$ be two finite closed intervals. Denote by $\overline{\mathcal{L}}(I,J)$ the set of left-continuous nondecreasing functions $f:I\to J$ such that $f(a)=A$ and $f(b)=B$. Dually, denote by $\widetilde{\mathcal{L}}(J,I)$ the set of right-continuous nondecreasing functions $g:J\to I$ such that $g(A)=a$ and $g(B)=b$. If $f\in\overline{\mathcal{L}}(I,J)$, its right-continuous inverse is defined as

$$\widetilde{f}(y)=\sup\{x\in I: f(x)\le y\},\quad y\in J. \tag{12}$$

The following auxiliary result will be needed. Although tedious, its proof is straightforward (for more details, see Adell and Pérez-Palomares [3] or Winter [13]).

Lemma 1 *If $f\in\overline{\mathcal{L}}(I,J)$, then $\widetilde{f}\in\widetilde{\mathcal{L}}(J,I)$. Moreover, $f(x)\le y$ if and only if $x\le\widetilde{f}(y)$, $x\in I$, $y\in J$.*

In particular,

$$x\le\widetilde{f}(f(x)),\quad x\in I.$$

This result will be applied as follows. Let $f\in C[0,1]$, $I=[0,1]$, and $J=[0,\omega(f;1)]$. Obviously, the function $\omega(f;\cdot)$ belongs to $\overline{\mathcal{L}}(I,J)$. Denote by $\widetilde{\omega}(f;\cdot)$ its right-continuous inverse, as defined in (12). Then, Lemma 1 tells us, in particular, that

$$\delta\le\widetilde{\omega}(f;\omega(f;\delta)),\quad 0\le\delta\le 1. \tag{13}$$

From now on, $\mathbb{Y}$ is a triangular array of random variables satisfying (4).

Theorem 1 *Let $f\in C[0,1]$ and let c be as in (3). If $M_n\xrightarrow{(P)}0$, then*

$$\|B_n(f,\mathbb{Y};x)-f(x)\|_\infty\xrightarrow{(P)}0.$$

In such a case, we have for any $0<\delta\le 1$

$$P\Big(\|B_n(f,\mathbb{Y};x)-f(x)\|_\infty>(1+c)\omega(f;\delta)\Big)\le P(M_n>\delta),\quad n\ge 1/\delta^2.$$

Proof . Using (1), (3), and the triangular inequality, we see that

$$
\begin{aligned}
|B_n(f,\mathbb{Y};x)-f(x)| &= \left|\sum_{k=0}^{n}\big(f(Y_{n,k})-f(x)\big)p_{n,k}(x)\right| \\
&\le \|B_n(f;x)-f(x)\|_\infty + \sum_{k=0}^{n}\omega\left(f;\left|Y_{n,k}-\frac{k}{n}\right|\right)p_{n,k}(x) \\
&\le c\omega\left(f;\frac{1}{\sqrt{n}}\right)+\omega(f;M_n). \qquad (14)
\end{aligned}
$$

Let $\epsilon \in (0,\omega(f;1)]$. Whenever

$$
\omega\left(f;\frac{1}{\sqrt{n}}\right)\le \epsilon, \qquad (15)
$$

we have from (14)

$$
P\Big(\|B_n(f,\mathbb{Y};x)-f(x)\|_\infty > (1+c)\epsilon\Big) \le P(\omega(f;M_n)>\epsilon) = P(M_n > \widetilde{\omega}(f;\epsilon)), \qquad (16)
$$

where the last inequality follows from Lemma 1 and the comments following it. This shows the first statement in Theorem 1. The second one follows by choosing $\epsilon=\omega(f;\delta)$, $\delta\ge 1/\sqrt{n}$, in (16), taking into account (13) and the fact that inequality (15) is obviously satisfied. The proof is complete. □

For the stochastic polynomials defined in (11), we obtain better estimates than those in Theorem 1.

Theorem 2 *Let $f\in C^2[0,1]$. If $M_n \xrightarrow{(P)} 0$, then*

$$
\|B_n^\star(f,\mathbb{Y};x)-f(x)\|_\infty \xrightarrow{(P)} 0.
$$

In such a case, we have for any $\epsilon>0$

$$
P\left(\|B_n^\star(f,\mathbb{Y};x)-f(x)\|_\infty > \frac{5}{4}\|f''\|_\infty\epsilon\right) \le P(M_n>\sqrt{\epsilon}), \quad n\ge 1/\epsilon.
$$

Proof . Recall that the centered second moment for the classical Bernstein polynomials is given by

$$
\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^2 p_{n,k}(x)=\frac{x(1-x)}{n}. \qquad (17)
$$

Hence, using Taylor's formula and the inequality

$$
(a+b)^2\le 2\left(a^2+b^2\right), \quad a,b\in\mathbb{R},
$$

we have

$$
\begin{aligned}
|B_n^\star(f,\mathbb{Y};x)-f(x)| &\le \frac{\|f''\|_\infty}{2}\sum_{k=0}^{n}(Y_{n,k}-x)^2 p_{n,k}(x) \\
&\le \|f''\|_\infty\left(\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^2 p_{n,k}(x)+\sum_{k=0}^{n}\left|Y_{n,k}-\frac{k}{n}\right|^2 p_{n,k}(x)\right) \\
&\le \|f''\|_\infty\left(\frac{1}{4n}+M_n^2\right). \qquad (18)
\end{aligned}
$$

For $n\ge 1/\epsilon$, this readily implies the result. □

To compare Theorem 1 and 2, suppose that $f \in C^2[0,1]$. In such a case, we have

$$\omega(f;\epsilon) \le \|f'\|_\infty \epsilon, \quad \epsilon > 0.$$

Hence, it follows from Theorem 1 that

$$P\Big(\|B_n(f,\mathbb{Y};x) - f(x)\|_\infty > (1+c)\|f'\|_\infty \epsilon\Big) \le P\big(M_n > \epsilon\big), \quad n \ge 1/\epsilon^2.$$

On the other hand, Theorem 2 gives us the upper bound $P\big(M_n > \sqrt{\epsilon}\big)$, which is obviously less than or equal to $P\big(M_n > \epsilon\big)$ for $0 < \epsilon \le 1$. Finally, notice that whenever $0 < \epsilon \le 1$, the rate of convergence of $P\big(M_n > \sqrt{\epsilon}\big)$, as $n \to \infty$, is, in general, much more faster than that of $P\big(M_n > \epsilon\big)$. This fact is illustrated by Lemma 2 below in the case of uniform order statistics.

3 The case of uniform order statistics

In this section, we assume that the triangular array $\mathbb{Y}$ of random variables is given in terms of uniform order statistics, as defined in (6) and (7). In first place, it may be of interest to note that we have from (5) and (7)

$$D_n(f;x) := \mathbb{E}B_n(f,\mathbb{Y};x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 f(\theta)p_{n,k}(\theta)d\theta,$$

where $\mathbb{E}$ stands for mathematical expectation. The positive linear operator D_n is known in the literature as the Bernstein-Durrmeyer operator, and was introduced independently by both Durrmeyer [7] and Lupas [8] (see also Adell and De la Cal [2] for a probabilistic approach). This is nothing else but an example of the way in which many deterministic models turn to be the expectation of their stochastic counterparts. The most representative one is given by the Bernstein polynomials themselves. Actually, we can write

$$B_n(f;x) = \mathbb{E}f\left(\frac{S_n(x)}{n}\right),$$

where the random variable $S_n(x)$ has the binomial law with parameters n and x, i.e.,

$$P(S_n(x) = k) = p_{n,k}(x), \quad k = 0,1,\ldots,n. \tag{19}$$

In the case at hand, the estimate of the tail probabilities $P(M_n > \epsilon)$ will be based on two facts. First, the generalized Chebyshev's inequality, which asserts that given a random variable X and a nondecreasing function $\varphi : (0,\infty) \to (0,\infty)$, we have

$$P(|X| > \epsilon) \le \frac{\mathbb{E}\varphi(|X|)}{\varphi(\epsilon)}, \quad \epsilon > 0. \tag{20}$$

Second, the uniform estimate for the moments of even order of the Bernstein polynomials recently shown by the authors [1]

$$\sup_{0\le\theta\le 1} \mathbb{E}\left|\frac{S_n(\theta)}{n} - \theta\right|^{2j} \le \frac{(2j)!}{6^j j! n^j}, \quad j \in \mathbb{N}_0. \tag{21}$$

Lemma 2 *Let $\epsilon > 0$ and $0 < r < 1$. Then,*

$$P\left(M_n > \epsilon\right) \leq \frac{n+1}{\sqrt{1-r}} \exp\left(-\frac{3r}{2} n\epsilon^2\right).$$

Proof . Let $\alpha > 0$ to be chosen later on. Observe that the event $\{M_n > \epsilon\}$ equals to $\cup_{k=0}^n \left\{\left|V_{n,k} - \frac{k}{n}\right| > \epsilon\right\}$. Thus, using Chebyshev's inequality (20) with $\varphi(x) = \exp(\alpha x)$, we have

$$P\left(M_n > \epsilon\right) \leq \sum_{k=0}^n P\left(\left|V_{n,k} - \frac{k}{n}\right|^2 > \epsilon^2\right) \leq \exp(-\alpha\epsilon^2) \sum_{k=0}^n \mathbb{E} \exp\left(\alpha \left|V_{n,k} - \frac{k}{n}\right|^2\right). \tag{22}$$

On the other hand, we have from (7), (19) and estimate (21)

$$\sum_{k=0}^n \mathbb{E} \exp\left(\alpha \left|V_{n,k} - \frac{k}{n}\right|^2\right) = (n+1) \int_0^1 \sum_{k=0}^n \exp\left(\alpha \left|\theta - \frac{k}{n}\right|^2\right) p_{n,k}(\theta) d\theta$$

$$= (n+1) \int_0^1 \mathbb{E} \exp\left(\alpha \left|\frac{S_n(\theta)}{n} - \theta\right|^2\right) d\theta = (n+1) \sum_{j=0}^\infty \frac{\alpha^j}{j!} \int_0^1 \mathbb{E} \left|\frac{S_n(\theta)}{n} - \theta\right|^{2j} d\theta$$

$$\leq (n+1) \sum_{j=0}^\infty \binom{2j}{j} \left(\frac{\alpha}{6n}\right)^j = (n+1) \sum_{j=0}^\infty \binom{-1/2}{j} \left(\frac{-4\alpha}{6n}\right)^j = (n+1) \left(1 - \frac{2\alpha}{3n}\right)^{-1/2}, \tag{23}$$

where the last two equalities follow from the identities

$$\binom{2j}{j} = \binom{-1/2}{j} (-4)^j, \quad j \in \mathbb{N}_0,$$

and

$$(1+x)^{-1/2} = \sum_{j=0}^\infty \binom{-1/2}{j} x^{-1/2}, \quad |x| < 1.$$

Choosing $\alpha = 3rn/2$, the result follows from (22) and (23). □

We are in a position to apply Theorem 1 and Theorem 2 to the case of triangular arrays based on uniform order statistics. To this end, let $(\tau(n))_{n\geq 1}$ be a sequence of real numbers satisfying the conditions

$$\lim_{n\to\infty} \tau(n) = \infty, \quad \lim_{n\to\infty} \frac{\tau(n)}{n} = 0, \quad \tau(n) \geq 1, \quad n \in \mathbb{N}. \tag{24}$$

Corollary 1 *Let $0 < r < 1$, and let c and $(\tau(n))_{n\geq 1}$ be as in (3) and (24), respectively. For any $f \in C[0,1]$ and $n \in \mathbb{N}$, we have*

$$P\left(\|B_n(f, \mathbb{Y}; x) - f(x)\|_\infty > (1+c)\omega\left(f; \sqrt{\frac{\tau(n)}{n}}\right)\right) \leq \frac{n+1}{\sqrt{1-r}} \exp\left(-\frac{3r}{2}\tau(n)\right). \tag{25}$$

Proof . It suffices to choose $\delta = \sqrt{\tau(n)/n}$ in Theorem 1 and to apply Lemma 2, by observing that $n \geq 1/\delta^2$, since $\tau(n) \geq 1$, thanks to assumption (24). □

The assumptions in (24) referring to the limiting behaviour of the sequence $(\tau(n))_{n\geq 1}$ are not used in the proof of Corollary 1. However, such assumptions are reasonable in view of inequality (25). In fact, Corollary 1 only has meaning if the right-hand side in (25) tends to 0, as $n\to\infty$. We could choose, for instance, $\tau(n)\sim\log(n+1)$ or $\tau(n)\sim n^{\alpha}$, $0<\alpha<1$. In making this choice, we need to balance the length of the confidence band, whose order of magnitude is $\omega(f;\sqrt{\tau(n)/n})$, and the speed of convergence to 0 of the right-hand side in (25).

To reformulate Corollary 1 in terms of conjecture (9), we give the following result.

Corollary 2 *Let $\epsilon>0$ and $a,r\in(0,1)$. Suppose that n is large enough so that*

$$\omega\left(f;\frac{1}{\sqrt{n}}\right)\leq\frac{2a}{3}\epsilon. \tag{26}$$

Then,

$$P\big(\|B_n(f,\mathbb{Y};x)-f(x)\|_\infty>\epsilon\big)\leq\frac{n+1}{\sqrt{1-r}}\exp\left(-\frac{3r}{2}(1-a)^2\frac{\epsilon^2}{\omega^2(f;1/\sqrt{n})}\right).$$

Proof . Directly from (5), and using the well known subadditivity property of the first modulus of continuity, we have

$$|B_n(f,\mathbb{Y};x)-f(x)|=\left|\sum_{k=0}^{n}\big(f(Y_{n,k})-f(x)\big)p_{n,k}(x)\right|$$

$$\leq\sum_{k=0}^{n}\omega(f;|Y_{n,k}-x|)p_{n,k}(x)\leq\omega\left(f;\frac{1}{\sqrt{n}}\right)\sum_{k=0}^{n}(1+\sqrt{n}\big|Y_{n,k}-x\big|)p_{n,k}(x)$$

$$\leq\omega\left(f;\frac{1}{\sqrt{n}}\right)\left(1+\sqrt{n}\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|p_{n,k}(x)+\sqrt{n}M_n\right). \tag{27}$$

Using (17) and Schwarz's inequality, we have from (26) and (27)

$$|B_n(f,\mathbb{Y};x)-f(x)|\leq a\epsilon+\sqrt{n}\omega(f;1/\sqrt{n})M_n.$$

We therefore have from Lemma 2

$$P\big(\|B_n(f,\mathbb{Y};x)-f(x)\|_\infty>\epsilon\big)\leq P\big(\sqrt{n}\omega(f;1/\sqrt{n})M_n>(1-a)\epsilon\big)$$

$$\leq\frac{n+1}{\sqrt{1-r}}\exp\left(-\frac{3r}{2}(1-a)^2\frac{\epsilon^2}{\omega^2(f;1/\sqrt{n})}\right).$$

The proof is complete. □

Apart from providing explicit constantes, Corollary 2 gives us better rates of convergence than those in (9). Actually, it suffices to choose the parameters a and r in such a way that

$$\frac{3r}{2}(1-a)^2\geq\frac{1}{4}.$$

Although similar, we think that the statements in Corollary 1 are more informative that those in Corollary 2.

For smooth functions, we give the following result for the variant of the stochastic Bernstein polynomials defined in (11).

Corollary 3 *Let $0 < r < 1$ and let $(\tau(n))_{n\geq 1}$ be as in (24). For any $f \in C^2[0,1]$ and $n \in \mathbb{N}$, we have*

$$P\left(\|B_n^\star(f,\mathbb{Y};x) - f(x)\|_\infty > \frac{5}{4}\|f''\|_\infty \frac{\tau(n)}{n}\right) \leq \frac{n+1}{\sqrt{1-r}} \exp\left(-\frac{3r}{2}\tau(n)\right).$$

Proof . Choose $\epsilon = \tau(n)/n$ in Theorem 2 and apply Lemma 2. □

We conclude by comparing the statements in Corollary 1 and Corollary 3 with the deterministic setting described in (3). Inequality (3) tells us that, with probability one, the nth Bernstein polynomial $B_n(f;x)$ is within the 'confidence' band $f(x) \pm c\omega(f;1/\sqrt{n})$, $x \in [0,1]$. According to Corollary 1, the nth stochastic Bernstein polynomial $B_n(f,\mathbb{Y};x)$ is within the confidence band $f(x) \pm (1+c)\omega(f;\sqrt{\tau(n)/n})$ with high probability (asymptotically one). An analogous statement holds true for the nth variant $B_n^*(f,\mathbb{Y};x)$ considered in Corollary 3. The main difference in this last case is that the length of the confidence band, i.e.,

$$\frac{5}{4}\|f''\|_\infty \frac{\tau(n)}{n}$$

is, asymptotically, much more shorter that that in Corollary 1.

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