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# Deriving the priority weights from incomplete hesitant fuzzy preference relations in group decision making

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ABSTRACT

The concept of hesitant fuzzy preference relation (HFPR) has been recently introduced to allow the decision makers (DMs) to provide several possible preference values over two alternatives. This paper introduces a new type of fuzzy preference structure, called incomplete HFPRs, to describe hesitant and incomplete evaluation information in the group decision making (GDM) process. Furthermore, we define the concept of multiplicative consistency incomplete HFPR and additive consistency incomplete HFPR, and then propose two goal programming models to derive the priority weights from an incomplete HFPR based on multiplicative consistency and additive consistency respectively. These two goal programming models are also extended to obtain the collective priority vector of several incomplete HFPRs. Finally, a numerical example and a practical application in strategy initiatives are provided to illustrate the validity and applicability of the proposed models.



## 1. Introduction

Since the introduction of fuzzy sets by Zadeh [45], several extensions and generalizations have been proposed (see Ref. [6]), including the intuitionistic fuzzy sets [5], interval-valued fuzzy sets [44], type-2 fuzzy sets [24], type *n* fuzzy sets [15] and fuzzy multisets [23]. Another extension of fuzzy sets is called hesitant fuzzy sets (HFSs), which were firstly introduced by Torra [31]. The motivation for introducing HFSs is that it is sometimes difficult to determine the membership of an element into a set, and in some circumstances, this difficulty is because there is a set of possible values.

HFSs are a new effective tool used to express human's hesitancy in daily life and have been receiving an increasing amount of attention in different areas, mainly in group decision making (GDM) [7,12,27,29,34,43,46]. Xia and Xu [33] defined the hesitant fuzzy preference relations (HFPRs) and hesitant multiplicative preference relations (HMPRs), which are based on the fuzzy preference relations and multiplicative preference relations, respectively. There are two more types of preference relations: interval-valued hesitant preference relations (IVHPRs) [7] and hesitant fuzzy linguistic preference relations (HFLPRs) [48] which are based on the hesitant fuzzy linguistic term sets [27,28]. Relationships of HFSs with other types of fuzzy sets can be found in [26] (see Section 5) and a historical overview of the fuzzy sets extensions analyzing their relationship can be found in [6].

The key motivating factors to introducing the concept of incomplete HFPR can be summarized as follows: (1) all of the aforementioned preference relations (HFPR, IVHPR, HMPR and HFLPR) do not consider the incomplete information. (2) In many real decision making problems, due to time pressure, lack of knowledge, and the DM's limited expertise related with the problem domain [1–4,8,10,17–19,37,40], the DMs may obtain a preference relation with incomplete entries. Incomplete HFPR do not merely permit the DMs to provide all of the possible values, but also allow them to give null values when comparing two alternatives. (3) It can enrich the theoretical system of preference relations. Zhang et al. [47] proposed two estimation procedures to estimate the missing information in an expert's incomplete HFPR, which are based on Xu et al.'s [40] models.

GDM problems consist in finding the best alternative(s) from a set of feasible ones according to the preference relations provided by a group of experts. In order to rank the alternatives, one direct method is to derive priorities from the group preference relations. Dong et al. [14] developed a framework to deal with the individual selection problem of the numerical scale and prioritization method in AHP. Dong and Herrera-Viedma [13] proposed a

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consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets in the decision making problems.

Up to now, there has been no investigation of deriving the priority weights from the incomplete HFPR. The aim of this paper is to propose some models to obtain priorities from incomplete HFPRs which are based on multiplicative consistency [9,11,30] and additive consistency [3,8,38] of fuzzy preference relations [19,30,38,41], respectively. As the DM gives a HFPR, each comparison has several values and the DM is hesitant on these values, we should abstract the most reasonable information from these values. That is we could derive the most consistent fuzzy preference relation from the HFPR to make decision. This is the main idea of the paper, and it is a new idea to deal with HFPR.

These models are programming models for multiplicative consistency incomplete HFPR and additive consistency incomplete HFPR respectively. Furthermore, we extend these programing models to obtain the collective priority vector of several incomplete HFPRs for the sake of application in GDM process. To show the potential of this proposal, we introduce two illustrative cases of study to show the effectiveness of the developed models.

The remained of this paper is organized as follows. Section 2 briefly reviews some basic knowledge on fuzzy preference relation, HFS and HFPR. Section 3 introduces the concepts of incomplete HFPR, acceptable incomplete HFPR, multiplicative consistent incomplete HFPR and additive consistency incomplete HFPR. In Section 4, we develop some new goal programming models to derive the priority weights from multiplicative consistency incomplete HFPR and additive consistency incomplete HFPR. Section 5 provides a numerical example and a case study in GDM concerning strategy initiatives showing validity and applicability of the proposed models. Some conclusions are pointed out in Section 6.

## 2. Preliminaries

In this section, we will give the definitions of fuzzy preference relation, hesitant fuzzy set, hesitant fuzzy element and hesitant fuzzy preference relation.

Denote $N=\{1,2,\ldots,n\}$, $M=\{1,2,\ldots,m\}$. Let $X=\{x_1,x_2,\ldots,x_n\}$ $(n\geq 2)$ be a finite set of alternatives, where $x_i$ denotes the $i$th alternative.

### 2.1. Fuzzy preference relation

**Definition 1** [20]**.** Let $R=(r_{ij})_{n\times n}$ be a preference relation, then $R$ is called a fuzzy preference relation, if

$$r_{ij}\in[0,1],\quad r_{ij}+r_{ji}=1,\quad r_{ii}=0.5 \ \text{for all } i,j\in N. \tag{1}$$

**Definition 2** [30]**.** Let $R=(r_{ij})_{n\times n}$ be a fuzzy preference relation, then $R$ is called a multiplicative consistency fuzzy preference relation, if the following multiplicative transitivity is satisfied:

$$r_{ik}r_{kj}r_{ji}=r_{ki}r_{jk}r_{ij} \quad \text{for all } i,j,k\in N. \tag{2}$$

**Definition 3** [9,30]**.** If $R=(r_{ij})_{n\times n}$ is a multiplicative consistency fuzzy preference relation, then such a preference relation is given by

$$r_{ij}=\frac{w_i}{w_i+w_j},\quad i,j\in N. \tag{3}$$

where $W=(w_1,w_2,\ldots,w_n)^T$ is the priority weighting vector for the fuzzy preference relation $R=(r_{ij})_{n\times n}$ and $\sum_{i=1}^n w_i=1$, $w_i=0$, $i\in N$.

**Definition 4** [30]**.** Let $R=(r_{ij})_{n\times n}$ be a fuzzy preference relation, then $R$ is called an additive consistency fuzzy preference relation, if the following additive transitivity is satisfied:

$$r_{ij}=r_{ik}-r_{jk}+0.5 \quad \text{for all } i,j,k\in N. \tag{4}$$

For the additive consistency fuzzy preference relation, there is a function between the element $r_{ij}$ and the weights $w_i$ and $w_j$. The function is obtained as follows.

**Lemma 1** [39]**.** *Let $R=(r_{ij})_{n\times n}$ be a fuzzy additive transitive preference relation, $W=(w_1,w_2,\ldots,w_n)^T$ be the corresponding weighting vector, where $0\leq w_i\leq 1$, then there exists a positive number $\beta$, and such a relation can be expressed as follows:*

$$r_{ij}=0.5+\beta(w_i-w_j). \tag{5}$$

**Remark 1.** Lemma 1 denotes that there is an explicit function relation between $r_{ij}$ and the ranking values $w_i$ and $w_j$. Chiclana et al. [11] constructed a similar relationship between the additive reciprocal preference relation and utility values. Tanino [30] first established the above correspondence where $\beta$ always equals to 0.5, but it was later shown that the correspondence is not always valid from different perspectives [16,21,22,35,36,39]. In the following, we will determine the value of $\beta$.

**Theorem 1.** *If the priority vector of the additive transitive perfectly consistency fuzzy preference relation $R$ is derived by normalizing rank aggregation method, then $\beta=\frac{n-1}{2}$.*

**Proof.** If the priority vector of the additive transitive perfectly consistency fuzzy preference relation $R$ is derived by normalizing rank aggregation method [35], then

$$w_i=\frac{\sum_{k=1}^n r_{ik}-0.5}{\sum_{i=1}^n\sum_{k=1,k\neq i}^n r_{ik}}=\frac{\sum_{k=1}^n r_{ik}-0.5}{\frac{n(n-1)}{2}},\quad i\in N. \tag{6}$$

$$w_j=\frac{\sum_{k=1}^n r_{jk}-0.5}{\sum_{i=1}^n\sum_{k=1,k\neq j}^n r_{jk}}=\frac{\sum_{k=1}^n r_{jk}-0.5}{\frac{n(n-1)}{2}},\quad i\in N. \tag{7}$$

Introducing Eqs. (6) and (7) into Eq. (5), then

$$\begin{aligned} r_{ij}&=\beta(w_i-w_j)+0.5\\ &=\beta\frac{\sum_{k=1}^n(r_{ik}-r_{jk})}{\frac{n(n-1)}{2}}+0.5\end{aligned}$$

Since

$$r_{ij}=r_{ik}-r_{jk}+0.5$$

then

$$r_{ij}=\beta\frac{\sum_{k=1}^n(r_{ij}-0.5)}{\frac{n(n-1)}{2}}+0.5=\beta\frac{nr_{ij}-n/2}{\frac{n(n-1)}{2}}+0.5 \tag{8}$$

So we can get $\beta=\frac{n-1}{2}$, which complete the proof. □

That is to say, the relationship between $r_{ij}$ and $w_i-w_j$ is:

$$r_{ij}=0.5+\frac{n-1}{2}(w_i-w_j). \tag{9}$$

**Remark 2.** In addition, due to the fact that $0<r_{ij}<1$, we have $0<0.5+\frac{n-2}{2}(w_i-w_j)<1$, that is $-1/(n-1)<w_i-w_j<1/(n-1)$.

### 2.2. Hesitant fuzzy set

Torra [31] originally developed the definition of hesitant fuzzy sets (HFSs) as follows.

**Definition 5.** [31,32]. Let $X$ be a reference set, an HFS on $X$ is defined in terms of a function $h_A(x)$ that returns a non-empty subset of [0,1] when it is applied to $X$, i.e.

$$A=\{\langle x,h_A(x)\rangle | x\in X\}. \tag{10}$$

where $h_A(x)$ is a set of some different values in [0,1], representing the possible membership degrees of the element $x\in X$ to $A$. $h_A(x)$ is called a hesitant fuzzy element (HFE), a basic unit of HFS.

### 2.3. Hesitant fuzzy preference relation

On the basis of HFSs and FPRs, Xia and Xu [33] introduced hesitant fuzzy preference relations (HFPRs) as follows.

**Definition 6 [33,49].** Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, then a HFPR $H$ on $X$ is represented by a matrix $H = (h_{ij})_{n\times n} \subset X \times X$, where $h_{ij} = \{\gamma_{ij}^l | l = 1, \ldots, \#h_{ij}\}$ ($\#h_{ij}$ is the number of values in $h_{ij}$) is an HFE indicating all the possible values of preference degrees of the alternative $x_i$ over $x_j$. For all $i, j = 1, 2, \ldots, n$, $h_{ij}$ should satisfy the following conditions:

$$\begin{cases} \gamma_{ij}^{\sigma(l)} + \gamma_{ji}^{\sigma(l)} = 1 \\ h_{ii} = \{0.5\} \\ \#h_{ij} = \#h_{ji} \end{cases} \tag{11}$$

where $\gamma_{ij}^{\sigma(l)}$ is the $l$th largest element in $h_{ij}$.

## 3. Incomplete hesitant fuzzy preference relation

As described in the introduction, incomplete FPRs do not merely permit the DMs to provide all of the possible values, but also allow them to give null values when comparing two alternatives. This is formally defined as follows.

**Definition 7.** Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, then an incomplete HFPR $H$ on $X$ is represented by a matrix $H = (h_{ij})_{n\times n} \subset X \times X$, for all known HFEs $h_{ij} = \{\gamma_{ij}^l | l = 1, \ldots, \#h_{ij}\}$ ($\#h_{ij}$ is the number of values in $h_{ij}$) indicate all the possible values of preference degrees of the alternative $x_i$ over $x_j$ and should satisfy the following conditions:

$$\begin{cases} \gamma_{ij}^{\sigma(l)} + \gamma_{ji}^{\sigma(l)} = 1 \\ h_{ii} = \{0.5\} \\ \#h_{ij} = \#h_{ji} \end{cases} \tag{12}$$

where $\gamma_{ij}^{\sigma(l)}$ is the $l$th largest element in $h_{ij}$.

For the convenience of computations, we construct an indication matrix $\Delta = (\delta_{ij})_{n\times n}$ [42] of the incomplete HFPR $H = (h_{ij})_{n\times n}$, where

$$\delta_{ij} = \begin{cases} 0, & h_{ij} = - \\ 1, & h_{ij} \neq - \end{cases}, \quad \text{and} \quad h_{ij} = - \text{indicates a missing HFE } h_{ij}.$$

It should be noted that when $\delta_{ij} = 1$ for all $i, j \in N$, incomplete HFPR becomes complete HFPR, indicating that the latter is a special case of the former.

Based on the concepts of multiplicative consistency fuzzy preference relation and additive consistency fuzzy preference relation, we will introduce the concept of multiplicative consistency incomplete HFPR and additive consistency incomplete HFPR in the following.

**Definition 8.** Let $H = (h_{ij})_{n\times n}$ be an incomplete HFPR, if the missing HFEs of $H$ can be determined by the known HFEs, then $H$ is called an acceptable incomplete HFPR, otherwise, $H$ is not an acceptable incomplete HFPR.

**Theorem 2.** *Let $H = (h_{ij})_{n\times n}$ be an incomplete HFPR, the necessary condition of acceptable incomplete HFPR $H$ is that there is at least one known HFE in each row or column of $H$ except for the diagonal HFE,* i.e. *it needs at least $(n-1)$ judgments.*

**Definition 9.** Let $H = (h_{ij})_{n\times n}$ be an incomplete HFPR, then $H$ is called a multiplicative consistency incomplete HFPR, if some of its HFEs cannot be given by the DM, which we denote by the symbol $-$, and the others can be provided by the DM, which satisfy

$$\frac{w_i}{w_i + w_j} = \gamma_{ij}^{\sigma(1)} \quad or \ldots or \quad \gamma_{ij}^{\sigma(\#h_{ij})}, \quad i, j \in N. \tag{13}$$

**Definition 10.** Let $H = (h_{ij})_{n\times n}$ be an incomplete HFPR, then $H$ is called an additive consistency incomplete HFPR, if some of its HFE cannot be given by the DM, which we denote by symbol $-$, and the others can be provided by the DM, which satisfy

$$\gamma_{ij}^{\sigma(1)} \quad or \ldots or \quad \gamma_{ij}^{\sigma(\#h_{ij})} = 0.5 + \frac{n-1}{2}(w_i - w_j), \quad i, j \in N. \tag{14}$$

## 4. Deriving the priority weights from incomplete HFPRs in GDM

Let's suppose a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, and a constructed incomplete HFPR $H = (h_{ij})_{n\times n}$, where $h_{ij} = \{\gamma_{ij}^l | l = 1, 2, \ldots, \#h_{ij}\}$. Since each element in $h_{ij}$ is a possible preference degree for the comparison of the alternative $x_i$ over $x_j$.

(1) By Eq. (13), the multiplicative consistency preference relation can be obtained by:

$$\delta_{ij}\frac{w_i}{w_i + w_j} = \delta_{ij}\left(\gamma_{ij}^{\sigma(1)} or \ldots or\ \gamma_{ij}^{\sigma(\#h_{ij})}\right), \quad i, j \in N. \tag{15}$$

where $\gamma_{ij}^{\sigma(l)}$ is the $l$th largest element in $h_{ij}$, $\#h_{ij}$ is the number of elements in $h_{ij}$ and $\delta_{ij} = \{\begin{smallmatrix} 0, & h_{ij}=- \\ 1, & h_{ij}\neq - \end{smallmatrix}$.

Let $S(\gamma_{ij}) = \gamma_{ij}^{\sigma(1)} or \ldots\ or\ \gamma_{ij}^{\sigma(\#h_{ij})}$, by Eq. (15), we have

$$\begin{aligned} &\delta_{ij}\frac{w_i}{w_i + w_j} = \delta_{ij}S(\gamma_{ij}) \\ &\Leftrightarrow \delta_{ij}w_i = \delta_{ij}(w_i + w_j)(S(\gamma_{ij})) \\ &\Leftrightarrow \delta_{ij}(1 - S(\gamma_{ij}))w_i = \delta_{ij}S(\gamma_{ij})w_j, i, j \in N. \end{aligned} \tag{16}$$

Due to the fact that $1 - S(\gamma_{ij}) = (1 - \gamma_{ij}^{\sigma(1)}) or \ldots or (1 - \gamma_{ij}^{\sigma(\#h_{ij})})$, then we have $1 - S(\gamma_{ij}) = S(\gamma_{ji})$, thus Eq. (16) can be rewritten as

$$\delta_{ij}S(\gamma_{ji})w_i = \delta_{ij}S(\gamma_{ij})w_j, \quad i, j \in N. \tag{17}$$

Nevertheless, Eq. (17) does not always hold in the general case. There is deviation between $\delta_{ij}S(\gamma_{ji})w_i$ and $\delta_{ij}S(\gamma_{ij})w_j$, and the deviation degree is given by Eq. (18).

$$\varepsilon_{ij} = \delta_{ij}|S(\gamma_{ji})w_i - S(\gamma_{ij})w_j| \tag{18}$$

Thus, we could construct the following multi-objective programming model:

$$\begin{aligned} (\text{M}-\mathbf{1}) \ \min \quad & \varepsilon_{ij} = \delta_{ij}|S(\gamma_{ji})w_i - S(\gamma_{ij})w_j|, \quad i, j \in N \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0, \quad i, j \in N. \end{aligned}$$

As $|S(\gamma_{ji})w_i - S(\gamma_{ij})w_j| = |S(\gamma_{ij})w_j - S(\gamma_{ji})w_i|$, the above minimization problem could be solved by solving the following programming model:

$$\begin{aligned} (\text{M}-2) \ \min \quad & F = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} s_{ij}d_{ij}^{+} + t_{ij}d_{ij}^{-} \\ \text{s.t.} \quad & \begin{cases} \delta_{ij}(S(\gamma_{ji})w_i - S(\gamma_{ij})w_j) - d_{ij}^{+} + d_{ij}^{-} = 0, \ \ i, j \in N, \ \ j > i \\ \sum_{i=1}^{n} w_i = 1, \ w_i \geq 0, \ \ i \in N \\ d_{ij}^{+}, d_{ij}^{-} \geq 0, \ \ i, j \in N, \ \ j > i \end{cases} \end{aligned}$$

where $d_{ij}^{+}$ is the positive deviation from the target of the goal $\varepsilon_{ij}$, defined as

$$d_{ij}^{+} = \delta_{ij}(S(\gamma_{ji})w_i - S(\gamma_{ij})w_j) \vee 0.$$

$d_{ij}^{-}$ is the negative deviation from the target of the goal $\varepsilon_{ij}$, defined as

$$d_{ij}^{-} = \delta_{ij}(S(\gamma_{ij})w_j - S(\gamma_{ji})w_i) \vee 0.$$

$s_{ij}$ and $t_{ij}$ are the weights corresponding to $d_{ij}^{+}$ and $d_{ij}^{-}$, respectively.

In order to solve the above problem, model (M-2) can be transformed into the following mixed 0-1 goal programming model:

$$(\mathrm{M{-}3})\ \min\ F=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} s_{ij}d_{ij}^{+}+t_{ij}d_{ij}^{-}$$

$$\text{s.t.}\begin{cases}\delta_{ij}\left[\left(\sum_{l=1}^{\#h_{ji}} z_{ji}^{\sigma(l)}\gamma_{ji}^{\sigma(l)}\right)w_i-\left(\sum_{l=1}^{\#h_{ji}} z_{ji}^{\sigma(l)}\gamma_{ij}^{\sigma(l)}\right)w_j\right]\\ \quad -d_{ij}^{+}+d_{ij}^{-}=0,\quad i,j\in N,\ j>i\\ \sum_{i=1}^{n} w_i=1,\quad w_i\geq 0,\quad i\in N\\ \sum_{l=1}^{\#h_{ji}} z_{ji}^{\sigma(l)}=1,\quad i,j\in N,\quad j>i\\ z_{ji}^{\sigma(l)}=0\ \text{ or }\ 1,\quad i,j\in N,\quad l=1,2,\ldots,\#h_{ji},\ j>i\\ d_{ij}^{+},\ d_{ij}^{-}\geq 0,\quad i,j\in N,\quad j>i\end{cases}$$

Without loss of generality, when we consider that all the goal functions $\varepsilon_{ij}(i,j\in N)$ are fair, then we can set $s_{ij}=t_{ij}=1\ (i,j\in N)$. Consequently, model (M-3) can be rewritten as follows.

$$(\mathrm{M{-}4})\ \min\ F=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(d_{ij}^{+}+d_{ij}^{-})$$

$$\text{s.t.}\begin{cases}\delta_{ij}\left[\left(\sum_{l=1}^{\#h_{ji}} z_{ji}^{\sigma(l)}\gamma_{ji}^{\sigma(l)}\right)w_i-\left(\sum_{l=1}^{\#h_{ji}} z_{ji}^{\sigma(l)}\gamma_{ij}^{\sigma(l)}\right)w_j\right]\\ \quad -d_{ij}^{+}+d_{ij}^{-}=0,\quad i,j\in N,\ j>i\\ \sum_{i=1}^{n} w_i=1,\quad w_i\geq 0,\quad i\in N\\ \sum_{l=1}^{\#h_{ji}} z_{ji}^{\sigma(l)}=1,\quad i,j\in N,\quad j>i\\ z_{ji}^{\sigma(l)}=0\ \text{ or }\ 1,\quad i,j\in N,j>i,\quad l=1,2,\ldots,\#h_{ji}\\ d_{ij}^{+},\ d_{ij}^{-}\geq 0,\quad i,j\in N,\ j>i\end{cases}$$

(2) By Eq. (14), the additive consistency preference relation can be obtained by:

$$\delta_{ij}\left(\gamma_{ij}^{\sigma(1)} or \ldots\ or\ \gamma_{ij}^{\sigma(\#h_{ij})}\right)=\delta_{ij}\left[0.5+\frac{n-1}{2}(w_i-w_j)\right],\quad i,j\in N \tag{19}$$

where $\gamma_{ij}^{\sigma(l)}$ is the $l$th largest element in $h_{ij}$, $\#h_{ij}$ is the number of elements in $H$ and $\delta_{ij}=\{\begin{smallmatrix}0, & h_{ij}=-\\ 1, & h_{ij}\neq -\end{smallmatrix}$.

Let $S(\gamma_{ij})=\gamma_{ij}^{(1)} or \ldots or\ \gamma_{ij}^{(\#h_{ij})}$, by Eq. (19), we have

$$\delta_{ij}S(\gamma_{ij})=\delta_{ij}\left[0.5+\frac{n-1}{2}(w_i-w_j)\right]$$

Let $\varepsilon_{ij}=\delta_{ij}|S(\gamma_{ij})-[0.5+\frac{n-1}{2}(w_i-w_j)]|$, to obtain as many additive consistency preferences as possible, we construct the following multi-objective programming model:

$$(\mathrm{M}-5)\ \min\ \varepsilon_{ij}=\delta_{ij}\left|S(\gamma_{ij})-\left[0.5+\frac{n-1}{2}(w_i-w_j)\right]\right|,\quad i,j\in N.$$

$$\text{s.t.}\ \sum_{i=1}^{n} w_i=1,\quad w_i\geq 0, i,j\in N.$$

The solution to the above minimization problem is found by solving the following goal programming model:

$$(\mathrm{M}-6)\ \min\ F=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} s_{ij}d_{ij}^{+}+t_{ij}d_{ij}^{-}$$

$$\text{s.t.}\begin{cases}\delta_{ij}\left[S(\gamma_{ij})-\left(0.5+\frac{n-1}{2}(w_i-w_j)\right)\right]\\ \quad -d_{ij}^{+}+d_{ij}^{-}=0,\quad i,j\in N,\ j>i\\ \sum_{i=1}^{n} w_i=1,\quad w_i\geq 0,\quad i\in N\\ d_{ij}^{+},\ d_{ij}^{-}\geq 0,\quad i,j\in N,\quad j>i\end{cases}$$

where $d_{ij}^{+}$ is the positive deviation from the target of the goal $\varepsilon_{ij}$, defined as

$$d_{ij}^{+}=\delta_{ij}\left[S(\gamma_{ij})-\left(0.5+\frac{n-1}{2}(w_i-w_j)\right)\right]\vee 0.$$

$d_{ij}^{-}$ is the negative deviation from the target of the goal $\varepsilon_{ij}$, defined as

$$d_{ij}^{-}=\delta_{ij}\left[\left(0.5+\frac{n-1}{2}(w_i-w_j)\right)-S(\gamma_{ij})\right]\vee 0.$$

$s_{ij}$ and $t_{ij}$ are the weights corresponding to $d_{ij}^{+}$ and $d_{ij}^{-}$, respectively.

Similarly, model (M-6) can be transformed into the following 0-1 mixed goal programming:

$$(\mathrm{M}-7)\ \min\ F=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} s_{ij}d_{ij}^{+}+t_{ij}d_{ij}^{-}$$

$$\text{s.t.}\begin{cases}\delta_{ij}\left[\sum_{l=1}^{\#h_{ij}} z_{ij}^{\sigma(l)}\gamma_{ij}^{\sigma(l)}-\left(0.5+\frac{n-1}{2}(w_i-w_j)\right)\right]\\ \quad -d_{ij}^{+}+d_{ij}^{-}=0,\quad i,j\in N,\quad j>i\\ \sum_{i=1}^{n} w_i=1,\quad w_i\geq 0,\quad i\in N\\ \sum_{l=1}^{\#h_{ij}} z_{ij}^{\sigma(l)}=1,\quad i,j\in N,\quad j>i\\ z_{ij}^{\sigma(l)}=0\ \text{ or }\ 1,\quad i,j\in N,\quad l=1,2,\ldots,\#h_{ij},\ j>i\\ d_{ij}^{+},\ d_{ij}^{-}\geq 0,\quad i,j\in N,\quad j>i\end{cases}$$

Without loss of generality, when we consider that all the goal functions $\varepsilon_{ij}\ (i,j\in N)$ are fair, then we can set $s_{ij}=t_{ij}=1\ (i,j\in N)$. Consequently, model (M-7) can be rewritten as follows.

$$(\mathrm{M}-8)\ \min\ F=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(d_{ij}^{+}+d_{ij}^{-})$$

$$\text{s.t.}\begin{cases}\delta_{ij}\left[\sum_{l=1}^{\#h_{ij}} z_{ij}^{\sigma(l)}\gamma_{ij}^{\sigma(l)}-\left(0.5+\frac{n-1}{2}(w_i-w_j)\right)\right]\\ \quad -d_{ij}^{+}+d_{ij}^{-}=0,\quad i,j\in N,\quad j>i\\ \sum_{i=1}^{n} w_i=1,\quad w_i\geq 0,\quad i\in N\\ \sum_{l=1}^{\#h_{ij}} z_{ij}^{\sigma(l)}=1,\quad i,j\in N,\quad j>i\\ z_{ij}^{\sigma(l)}=0\ \text{ or }\ 1,\quad i,j\in N,\quad j>i, l=1,2,\ldots,\#h_{ij}\\ d_{ij}^{+},\ d_{ij}^{-}\geq 0,\quad i,j\in N,\quad j>i\end{cases}$$

Once we have developed an algorithm for solving a computational problem and analyzed its worst-case time requirements as a function of the size of its input (in terms of the O-notation). We analyze the complexity of computation of different models. Time complexity of (M-4) –(M-8) is $o(n^4\times l)$, which depends on product of $n^4$ ($n$ is the number of alternatives) and $l$ ($l = \prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \#h_{ij}$, $i, j \in N$, $j > i$). Meanwhile, the space complexity of (M-4) –(M-8) is $o(1)$.

By solving this model, we can also obtain the priority vector $W = (w_1, w_2, \ldots, w_n)^T$ of the incomplete HFPR $H = (h_{ij})_{n\times n}$. We will extend the above models to obtain the collective priority vector of two or more incomplete HFPRs.

Suppose that there are $m$ incomplete HFPRs $H_k = (h_{ij,k})_{n\times n}(k \in M)$, and $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$ is their collective priority vector, where $\nu_i \geq 0$, $i \in N$, $\sum_{i=1}^{n} \nu_i = 1$. $E = \{e_1, e_2, \ldots, e_m\}$ be a finite set of experts, where $e_k$ denotes the $k$th expert. $U = (u_1, u_2, \ldots, u_m)^T$ be the weighting vector of experts, where $\sum_{k=1}^{m} u_k = 1$, $u_k \geq 0$ and $u_k$ means the importance degree of expert $e_k$. We also construct $m$ indication matrices $\Delta_k = (\delta_{ij,k})_{n\times n}$ $(k \in M)$ of the incomplete HFPRs $H_k = (h_{ij,k})_{n\times n}$ $(k \in M)$, where

$$\delta_{ij,k} = \begin{cases} 0, & h_{ij,k} = - \\ 1, & h_{ij,k} \neq - \end{cases}$$

For the multiplicative consistency HFPR, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$ can be obtained by solving the following model, which is an extension of the model (M-4):

$$(\text{M}-9)\ \min\ F = \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} u_k\left(d_{ij,k}^{+} + d_{ij,k}^{-}\right)$$

$$\text{s.t.}\begin{cases} \delta_{ij,k}(S(\gamma_{ji,k})\nu_i - S(\gamma_{ij,k})\nu_j) - d_{ij,k}^{+} + d_{ij,k}^{-} = 0, & i, j \in N, j > i \\ \sum_{i=1}^{n} \nu_i = 1, \quad \nu_i \geq 0, \quad i \in N \\ d_{ij,k}^{+}, \quad d_{ij,k}^{-} \geq 0, \quad i, j \in N, j > i, \quad k \in M \end{cases}$$

where $d_{ij,k}^{+}$ is the positive deviation, defined as

$$d_{ij,k}^{+} = \delta_{ij,k}(S(\gamma_{ji,k})\nu_i - S(\gamma_{ij,k})\nu_j) \vee 0.$$

$d_{ij,k}^{-}$ is the negative deviation, defined as

$$d_{ij,k}^{-} = \delta_{ij,k}(S(\gamma_{ij,k})\nu_j - S(\gamma_{ji,k})\nu_i) \vee 0.$$

Model (M-9) can be transformed into the following 0-1 mixed goal programming:

$$(\text{M}-10)\ \min\ F = \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} u_k\left(d_{ij,k}^{+} + d_{ij,k}^{-}\right)$$

$$\text{s.t.}\begin{cases} \delta_{ij}^{(k)}\left[\left(\sum_{l=1}^{\#h_{ji,k}} z_{ji,k}^{\sigma(l)}\gamma_{ji,k}^{\sigma(l)}\right)\nu_i - \left(\sum_{l=1}^{\#h_{ji,k}} z_{ji,k}^{\sigma(l)}\gamma_{ij,k}^{\sigma(l)}\right)\nu_j\right] \\ \qquad -d_{ij,k}^{+} + d_{ij,k}^{-} = 0, \quad i, j \in N, j > i, k \in M \\ \sum_{i=1}^{n} \nu_i = 1, \quad \nu_i \geq 0, \quad i \in N \\ \sum_{l=1}^{\#h_{ji,k}} z_{ji,k}^{\sigma(l)} = 1, \quad i, j \in N, \quad j > i, k \in M \\ z_{ji,k}^{\sigma(l)} = 0 \text{ or } 1, \quad i, j \in N, \quad j > i, k \in M, \quad l = 1, 2, \ldots, \#h_{ji,k} \\ d_{ij,k}^{+}, \ d_{ij,k}^{-} \geq 0, \quad i, j \in N, \quad j > i, \quad k \in M \end{cases}$$

For the additive consistency HFPR, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$ can be obtained by solving the following model, which is an extension of the model (M-8):

$$(\text{M}-11)\ \min\ F = \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} u_k\left(d_{ij,k}^{+} + d_{ij,k}^{-}\right)$$

$$\text{s.t.}\begin{cases} \delta_{ij,k}\left[S(\gamma_{ij,k}) - \left(0.5 + \frac{n-1}{2}(\nu_i - \nu_j)\right)\right] \\ \qquad -d_{ij,k}^{+} + d_{ij,k}^{-} = 0, \quad i, j \in N, \quad j > i \\ \sum_{i=1}^{n} \nu_i = 1, \quad \nu_i \geq 0, \quad i \in N \\ d_{ij,k}^{+}, \quad d_{ij,k}^{-} \geq 0, \quad i, j \in N, \quad j > i, \quad k \in M \end{cases}$$

where $d_{ij,k}^{+}$ is the positive deviation, defined as

$$d_{ij,k}^{+} = \delta_{ij,k}\left[S(\gamma_{ij,k}) - \left(0.5 + \frac{n-1}{2}(\nu_i - \nu_j)\right)\right] \vee 0.$$

$d_{ij,k}^{-}$ is the negative deviation, defined as

$$d_{ij,k}^{-} = \delta_{ij,k}\left[\left(0.5 + \frac{n-1}{2}(\nu_i - \nu_j)\right) - S(\gamma_{ij,k})\right] \vee 0$$

Model (M-11) can be transformed into the following 0-1 mixed goal programming:

$$(\text{M}-12)\ \min\ F = \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} u_k\left(d_{ij,k}^{+} + d_{ij,k}^{-}\right)$$

$$\text{s.t.}\begin{cases} \delta_{ij,k}\left[\sum_{l=1}^{\#h_{ij,k}} z_{ij,k}^{\sigma(l)}\gamma_{ij,k}^{\sigma(l)} - \left(0.5 + \frac{(n-1)}{2}(\nu_i - \nu_j)\right)\right] \\ \qquad -d_{ij,k}^{+} + d_{ij,k}^{-} = 0, \quad i, j \in N, \quad j > i \\ \sum_{i=1}^{n} \nu_i = 1, \quad \nu_i \geq 0, \quad i \in N \\ \sum_{l=1}^{\#h_{ij,k}} z_{ij,k}^{\sigma(l)} = 1, \quad i, j \in N, \quad j > i, \quad k \in M \\ d_{ij,k}^{+}, \ d_{ij,k}^{-} \geq 0, \quad i, j \in N, \quad j > i, \quad k \in M \end{cases}$$

## 5. Illustrative cases of study

In this section, two numerical examples are provided to demonstrate the practicality and effectiveness of the developed models.

### 5.1. Case of study with four decision alternatives and an incomplete HFPR

Consider a single DM's decision problem with four alternatives $x_i$ $(i = 1, 2, 3, 4)$. The DM provides his/her preferences over the four decision alternatives, as an incomplete HFPR as follows:

$$H = \begin{bmatrix} \{0.5\} & - & \{0.6, 0.7\} & \{0.4\} \\ - & \{0.5\} & \{0.4\} & - \\ \{0.4, 0.3\} & \{0.6\} & \{0.5\} & \{0.3, 0.4\} \\ \{0.6\} & - & \{0.7, 0.6\} & \{0.5\} \end{bmatrix}.$$

Based on Theorem 2, we know that $H$ is an acceptable incomplete HFPR, which means that the priority weights can be derived by the known HFEs.

(1) According to the model (M-4), we can construct the following 0-1 goal programming model:

$$\min\ F = \sum_{i=1}^{3}\sum_{j=i+1}^{4}(d_{ij}^{+} + d_{ij}^{-})$$

$$\text{s.t.}\begin{cases} -d_{12}^{+}+d_{12}^{-}=0\\ \left(z_{31}^{\sigma(1)}\times 0.4+z_{31}^{\sigma(2)}\times 0.3\right)w_1\\ \quad -\left(z_{31}^{\sigma(1)}\times 0.6+z_{31}^{\sigma(2)}\times 0.7\right)w_3-d_{13}^{+}+d_{13}^{-}=0\\ 0.6w_1-0.4w_4-d_{14}^{+}+d_{14}^{-}=0\\ 0.6w_2-0.4w_3-d_{23}^{+}+d_{23}^{-}=0\\ \left(z_{43}^{\sigma(1)}\times 0.7+z_{43}^{\sigma(2)}\times 0.6\right)w_3\\ \quad -\left(z_{43}^{\sigma(1)}\times 0.3+z_{43}^{\sigma(2)}\times 0.4\right)w_4-d_{34}^{+}+d_{34}^{-}=0\\ w_1+w_2+w_3+w_4=1\\ z_{31}^{\sigma(1)}+z_{31}^{\sigma(2)}=1\\ z_{43}^{\sigma(1)}+z_{43}^{\sigma(2)}=1\\ z_{31}^{\sigma(1)},\ \ z_{31}^{\sigma(2)},\ \ z_{43}^{\sigma(1)},\ \ z_{43}^{\sigma(2)}=0\ \text{ or }\ 1\\ d_{ij}^{+},\ \ d_{ij}^{-}\geq 0,\ \ i,j=1,2,3,4,\ \ j>i \end{cases}$$

By solving the above optimization problem, we have:

$$F=0.004,\quad w_1=0.28,\quad w_2=0.12,\quad w_3=0.18,$$
$$w_4=0.42,\quad d_{12}^{+}=d_{12}^{-}=0,$$
$$d_{13}^{+}=0.004, d_{13}^{-}=0,\quad d_{23}^{+}=d_{23}^{-}=0,\quad d_{14}^{+}=d_{14}^{-}=0,$$
$$d_{24}^{+}=d_{24}^{-}=0,\quad d_{34}^{+}=d_{34}^{-}=0.$$

Therefore, the ranking of these four alternatives is $x_4\succ x_1\succ x_3\succ x_2$.

(2) According to the model (M-8), we can build this optimization problem as follows:

$$\min\ \ F=\sum_{i=1}^{3}\sum_{j=i+1}^{4}(d_{ij}^{+}+d_{ij}^{-})$$

$$\text{s.t.}\begin{cases} -d_{12}^{+}+d_{12}^{-}=0\\ z_{13}^{\sigma(1)}\times 0.6+z_{13}^{\sigma(2)}\times 0.7\\ \quad -(0.5+1.5(w_1-w_3))-d_{13}^{+}+d_{13}^{-}=0\\ 0.4-(0.5+1.5(w_1-w_4))-d_{14}^{+}+d_{14}^{-}=0\\ 0.4-(0.5+1.5(w_2-w_3))-d_{23}^{+}+d_{23}^{-}=0\\ -d_{24}^{+}+d_{24}^{-}=0\\ z_{34}^{\sigma(1)}\times 0.3+z_{34}^{\sigma(2)}\times 0.4\\ \quad -(0.5+1.5(w_3-w_4))-d_{34}^{+}+d_{34}^{-}=0\\ w_1+w_2+w_3+w_4=1\\ z_{13}^{\sigma(1)}+z_{13}^{\sigma(2)}=1\\ z_{34}^{\sigma(1)}+z_{34}^{\sigma(2)}=1\\ z_{13}^{\sigma(1)},\ \ z_{13}^{\sigma(2)},\ \ z_{34}^{\sigma(1)},\ \ z_{34}^{\sigma(2)}=0\ \text{ or }\ 1\\ d_{ij}^{+},\ d_{ij}^{-}\geq 0,\ \ i,j=1,2,3,4,\ \ j>i \end{cases}$$

By solving the above optimization problem, we have:

$$F=0,\quad w_1=0.2833,\quad w_2=0.1500,\quad w_3=0.2167,$$
$$w_4=0.3500,\quad d_{12}^{+}=d_{12}^{-}=0,$$
$$d_{13}^{+}=d_{13}^{-}=0,\quad d_{23}^{+}=d_{23}^{-}=0,\quad d_{14}^{+}=d_{14}^{-}=0,$$
$$d_{24}^{+}=d_{24}^{-}=0,\quad d_{34}^{+}=d_{34}^{-}=0.$$

So the ranking of these four alternatives is $x_4\succ x_1\succ x_3\succ x_2$, which is same as that obtained by the model (M-4).

To further compare the performances of these two models in fitting incomplete HFPRs, the following evaluation criteria are introduced:

Maximum Deviation (MD) for incomplete HFPR

$$\text{MD}=\max_{i,j,k}\left\{\delta_{ij,k}\left(\frac{\gamma'_{ij,k}}{\gamma'_{ji,k}}\frac{w_j}{w_i}+\frac{\gamma'_{ji,k}}{\gamma'_{ij,k}}\frac{w_i}{w_j}-2\right)\middle|\, i,j\in N,k\in M\right\} \tag{20}$$

**Table 1**
Performance comparisons for Example 1.

| Methods | $W^*$ | Ranking | MD | MAD |
|---|---|---|---|---|
| Model (M-4) | $(0.2800, 0.1200, 0.1800, 0.4200)^T$ | $x_4\succ x_1\succ x_3\succ x_2$ | 0.0013 | 0.0087 |
| Model (M-8) | $(0.2833, 0.1500, 0.2167, 0.3500)^T$ | $x_4\succ x_1\succ x_3\succ x_2$ | 0.1369 | 0.0824 |

where $\gamma'_{ij,k}$ is the value of $S(\gamma_{ij,k})$ when the $F$ gets the minimal value.

Maximum Absolute Deviation (MAD) for incomplete HFPR

$$\text{MAD}=\max_{i,j,k}\left\{\delta_{ij,k}\left|\gamma'_{ij,k}-\frac{w_i}{w_i+w_j}\right|\middle|\, i,j\in N,k\in M\right\} \tag{21}$$

where $d_{ij,k}=\gamma'_{ij,k}-w_i/(w_i+w_j)$ is the fitting error for $\gamma'_{ij,k}$. If the priority vector $W=(w_1,\dots,w_n)^T$ is able to precisely fit the incomplete HFPR $H_k$, then $|d_{ij,k}|\equiv 0$, otherwise, $|d_{ij,k}|>0$.

From Table 1, it's shown that the model (M-4) achieves an identical ranking as model (M-8). Model (M-4) performs better than model (M-8) in terms of two performance evaluation criteria: MD and MAD, which partly shows the advantage of the model (M-4).

### 5.2. GDM problem with three alternatives and three experts

In the following, we further illustrate the practicality of incomplete HFPRs in group decision making by utilizing a practical example (adapted from [25]).

The enterprise's board of directors, which includes three members $e_k$ $(k=1,2,3)$, have to plan the development of large projects (strategy initiatives) for the following five years. Suppose that there are three possible projects $x_i$ $(i=1,2,3)$ to be evaluated. It is necessary to compare these projects in order to select that which is the most important as well as order them from the point view of their importance, taking into account four criteria suggested by the Balanced Scored methodology: (1) financial perspective; (2) the customer satisfaction; (3) internal business process perspective; (4) learning and growth perspective. First, the specialists are asked to give their opinion relative to each project. Because of the uncertainty of the attributes, it is difficult for the DMs to use just one value to provide their preferences. To facilitate the elicitation of their preferences, HFS is just an effective tool to deal with such situations. Furthermore, some experts may be lacking in knowledge and have limited expertise related to the problem domain, and thus, these members give their incomplete HFPRs as follows:

$$H_1=\begin{bmatrix}\{0.5\} & \{0.6\} & -\\ \{0.4\} & \{0.5\} & \{0.2,0.3\}\\ - & \{0.8,0.7\} & \{0.5\}\end{bmatrix},$$

$$H_2=\begin{bmatrix}\{0.5\} & - & \{0.3,0.4\}\\ - & \{0.5\} & \{0.3\}\\ \{0.7,0.6\} & \{0.7\} & \{0.5\}\end{bmatrix},$$

$$H_3=\begin{bmatrix}\{0.5\} & \{0.3,0.4\} & \{0.4\}\\ \{0.7,0.6\} & \{0.5\} & -\\ \{0.6\} & - & \{0.5\}\end{bmatrix}.$$

From Theorem 2, we know that $H_k$ $(k=1, 2, 3)$ are all acceptable incomplete HFPRs. That is, the priority vector can be obtained through the known HFEs. Without loss of generality, we set $u_1=u_2=u_3=1/3$.

(1) According to model (M-10), we can build this optimization problem as follows:

$$\min\ \ F=\sum_{k=1}^{3}\sum_{i=1}^{2}\sum_{j=i+1}^{3}u_k\left(d_{ij,k}^{+}+d_{ij,k}^{-}\right)$$

$$\text{s. t.}\begin{cases} 0.4v_1 - 0.6v_2 - d^+_{12,1} + d^-_{12,1} = 0 \\ \quad -d^+_{13,1} + d^-_{13,1} = 0 \\ \left(z^{\sigma(1)}_{32,1} \times 0.8 + z^{\sigma(2)}_{32,1} \times 0.7\right)v_2 - \left(z^{\sigma(1)}_{32,1} \times 0.2 + z^{\sigma(2)}_{32,1} \times 0.3\right)v_3 \\ \quad -d^+_{23,1} + d^-_{23,1} = 0 \\ \quad -d^+_{12,2} + d^-_{12,2} = 0 \\ \left(z^{\sigma(1)}_{31,2} \times 0.7 + z^{\sigma(2)}_{31,2} \times 0.6\right)v_1 - \left(z^{\sigma(1)}_{31,2} \times 0.3 + z^{\sigma(2)}_{31,2} \times 0.4\right)v_3 \\ \quad -d^+_{13,2} + d^-_{13,2} = 0 \\ 0.7v_2 - 0.3v_3 - d^+_{23,2} + d^-_{23,2} = 0 \\ \left(z^{\sigma(1)}_{21,3} \times 0.7 + z^{\sigma(2)}_{21,3} \times 0.6\right)v_1 - \left(z^{\sigma(1)}_{21,3} \times 0.3 + z^{\sigma(2)}_{21,3} \times 0.4\right)v_2 \\ \quad -d^+_{12,3} + d^-_{12,3} = 0 \\ 0.6v_1 - 0.4v_3 - d^+_{13,3} + d^-_{13,3} = 0 \\ \quad -d^+_{23,3} + d^-_{23,3} = 0 \\ v_1 + v_2 + v_3 = 1 \\ z^{\sigma(1)}_{32,1} + z^{\sigma(2)}_{32,1} = 1 \\ z^{\sigma(1)}_{31,2} + z^{\sigma(2)}_{31,2} = 1 \\ z^{\sigma(1)}_{21,3} + z^{\sigma(2)}_{21,3} = 1 \\ z^{\sigma(1)}_{32,1},\ z^{\sigma(2)}_{32,1},\ z^{\sigma(1)}_{31,2},\ z^{\sigma(2)}_{31,2},\ z^{\sigma(1)}_{21,3},\ z^{\sigma(2)}_{21,3} = 0 \text{ or } 1 \\ d^+_{ij,k}, d^-_{ij,k} \ge 0,\quad i,j = 1,2,3,\quad j > i,\quad k = 1,2,3 \end{cases}$$

By solving the above optimization problem, we have:

$F = 0.0379,\ \ v_1 = 0.3182,\ \ v_2 = 0.2045,\ \ v_3 = 0.4773,$
$d^+_{12,1} = 0.45 \times 10^{-2},$
$d^-_{12,1} = 0,\ \ d^+_{13,1} = d^-_{13,1} = 0,\ \ d^+_{23,1} = d^-_{23,1} = 0,\ \ d^+_{12,2} = d^-_{12,2} = 0,$
$d^+_{13,2} = d^-_{13,2} = 0$
$d^+_{23,2} = d^-_{23,2} = 0,\ \ d^+_{12,3} = 0.1091,\ \ d^-_{12,3} = 0,\ \ d^+_{13,3} = d^-_{13,3} = 0,$
$d^+_{23,3} = d^-_{23,3} = 0.$

Therefore, the ranking of these three alternatives is $x_3 \succ x_1 \succ x_2$.

(2) According to the model (M-12), we can build this optimization problem as follows:

$$\min\ F = \sum_{k=1}^{3}\sum_{i=1}^{2}\sum_{j=i+1}^{3} u_k\left(d^+_{ij,k} + d^-_{ij,k}\right)$$

$$\text{s.t.}\begin{cases} 0.6 - (0.5 + v_1 - v_2) - d^+_{12,1} + d^-_{12,1} = 0 \\ -d^+_{13,1} + d^-_{13,1} = 0 \\ \left[z^{\sigma(1)}_{23,1} \times 0.2 + z^{\sigma(2)}_{23,1} \times 0.3 - (0.5 + v_2 - v_3)\right] - d^+_{23,1} + d^-_{23,1} = 0 \\ -d^+_{12,2} + d^-_{12,2} = 0 \\ \left[z^{\sigma(1)}_{13,2} \times 0.3 + z^{\sigma(2)}_{13,2} \times 0.4 - (0.5 + v_1 - v_3)\right] - d^+_{13,2} + d^-_{13,2} = 0 \\ 0.3 - (0.5 + v_2 - v_3) - d^+_{23,2} + d^-_{23,2} = 0 \\ \left[z^{\sigma(1)}_{12,3} \times 0.3 + z^{\sigma(2)}_{12,3} \times 0.4 - (0.5 + v_1 - v_2)\right] - d^+_{12,3} + d^-_{12,3} = 0 \\ 0.4 - (0.5 + v_1 - v_3) - d^+_{13,3} + d^-_{13,3} = 0 \\ -d^+_{23,3} + d^-_{23,3} = 0 \\ v_1 + v_2 + v_3 = 1 \\ z^{\sigma(1)}_{23,1} + z^{\sigma(2)}_{23,1} = 1 \\ z^{\sigma(1)}_{13,2} + z^{\sigma(2)}_{13,2} = 1 \\ z^{\sigma(1)}_{12,3} + z^{\sigma(2)}_{12,3} = 1 \\ z^{\sigma(1)}_{23,1},\ z^{\sigma(2)}_{23,1},\ z^{\sigma(1)}_{13,2},\ z^{\sigma(2)}_{13,2},\ z^{\sigma(1)}_{12,3},\ z^{\sigma(2)}_{12,3} = 0 \text{ or } 1 \\ d^+_{ij,k},\ \ d^-_{ij,k} \ge 0,\quad i,j = 1,2,3,\quad j > i,\quad k = 1,2,3 \end{cases}$$

By solving the above optimization problem, we have:

$F = 0.0667,\ \ v_1 = 0.3333,\ \ v_2 = 0.2333,\ \ v_3 = 0.4333,$

**Table 2**
Performance comparisons for Example 2.

| Methods | $W^*$ | Ranking | MD | MAD |
|---|---|---|---|---|
| Model (M-10) | $(0.3182, 0.2045, 0.4773)^T$ | $x_3 \succ x_1 \succ x_2$ | 0.0013 | 0.0088 |
| Model (M-12) | $(0.3333, 0.2333, 0.4333)^T$ | $x_3 \succ x_1 \succ x_2$ | 0.0523 | 0.0500 |

$d^+_{12,1} = d^-_{12,1} = 0,$
$d^+_{13,1} = d^-_{13,1} = 0,\ \ d^+_{23,1} = d^-_{23,1} = 0,\ \ d^+_{12,2} = d^-_{12,2} = 0,$
$d^+_{13,2} = d^-_{13,2} = 0,$
$d^+_{23,2} = d^-_{23,2} = 0,\ \ d^+_{12,3} = 0.2,\ \ d^-_{12,3} = 0,\ \ d^+_{13,3} = d^-_{13,3} = 0,$
$d^+_{23,3} = d^-_{23,3} = 0.$

So the ranking of these three alternatives is $x_3 \succ x_1 \succ x_2$, which is same as that obtained by the model (M-10).

The results of comparisons are shown in Table 2, from which we can see that model (M-10) achieves the same ranking $x_3 \succ x_1 \succ x_2$ as model (M-12). Moreover, model (M-10) has smaller MD and MAD than model (M-12).

**Remark 2.** It should be noted that many methods have been proposed to derive the weighting vector for multiplicative preference relations and to solve group decision making problems. However, these methods fail when addressing situations in which the input arguments take the form of HFPRs. Because of the models are specifically used for HFPRs, and according to our knowledge, there is no previous work which concentrates on deriving the weighing vector from the incomplete HFPRs, it is not easy to continue with comparative analysis. In the following, we discuss some advantages and differences as compared with the existing different kinds of methods for GDM problems.

(1) In the above case study, Parreiras et al. [25] proposes a flexible consensus scheme for GDM problems under linguistic assessments. However, in their approach, they first transformed the linguistic variables into triangular fuzzy numbers, which led to information losing. Second, they used the consensus scheme to rank the alternatives, which is very complicated, while our method can rank the alternatives directly.
(2) For the incomplete FPRs, for example, Xu [42], Xu et al. [36], their methods did not consider the hesitant situation, which limits their application. However, if there is only one value in each pairwise value in the HFPRs, the methods proposed in this paper will become reduced to the traditional incomplete FPRs.
(3) Zhu et al. [50] presented the ranking methods with HFPRs in GDM environments. These methods only consider the multiplicative consistency of HFPRs. Generally, the cardinal consistency of FPRs includes multiplicative consistency and additive consistency. In this paper, we consider these different types of consistencies for HFPRs. Furthermore, if $\delta_{ij} = 1$ for all $i, j \in N$ in all the models (M-1)-(M-12), then the proposed methods can be used to derive the rankings for the complete HFPRs. Which means that the proposed methods can deal with both the complete and incomplete HFPRs, while Zhu et al.'s [45] method is only suitable to deal with the complete ones. In other words, Zhu et al.'s method would be considered as a special case of the proposed method.

## 6. Conclusions

We have investigated group decision making problems, where preference information offered by DMs is hesitant and incomplete. For the sake of a better description of this situation, we have proposed a new concept of incomplete HFPRs, which are an effective tool to collect and present preferences provided by DMs in decision making. Incomplete HFPRs do not merely permit the DMs to provide all of the possible values but also allow them to give null values when comparing two alternatives. In this paper, we also introduced the concept of multiplicative consistency incomplete HFPR and additive consistency incomplete HFPR. Moreover, to obtain the priority vector of an incomplete HFPR, we have proposed two programming models based on multiplicative consistency and additive consistency respectively. These two goal programming models are also extended to obtain the collective priority vector of several incomplete HFPRs. Finally, the practicability and effectiveness of the developed models have been verified using two illustrative examples.

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## References

[1] S. Alonso, F.J. Cabrerizo, F. Chiclana, F. Herrera, E. Herrera-Viedma, Group decision making with incomplete fuzzy linguistic preference relations, Int. J. Intell. Syst. 24 (2009) 201–222.
[2] S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, A learning procedure to estimate missing values in fuzzy preference relations based on additive consistency, Modeling Decisions for Artificial Intelligence, Springer, 2004, pp. 227–238.
[3] S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, J. Alcalá-Fdez, C. Porcel, A consistency-based procedure to estimate missing pairwise preference values, Int. J. Intell. Syst. 23 (2008) 155–175.
[4] S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera, A web based consensus support system for group decision making problems and incomplete preferences, Inf. Sci. 180 (2010) 4477–4495.
[5] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1986) 87–96.
[6] H. Bustince, E. Barrenechea, M. Pagola, J. Fernandez, Z.S. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera, B. De Baets, A historical account of types of fuzzy sets and their relationships, IEEE Trans. Fuzzy Syst. 24 (1) (2016) 179–194.
[7] N. Chen, Z.S. Xu, M.M. Xia, Interval-valued hesitant preference relations and their applications to group decision making, Knowl. Based Syst. 37 (2013) 528–540.
[8] F. Chiclana, E. Herrera-Viedma, S. Alonso, A note on two methods for estimating missing pairwise preference values, IEEE Trans. Syst. Man Cybern. Part B Cybern. 39 (2009) 1628–1633.
[9] F. Chiclana, E. Herrera-Viedma, S. Alonso, F. Herrera, Cardinal consistency of reciprocal preference relations: a characterization of multiplicative transitivity, IEEE Trans. Fuzzy Syst. 17 (2009) 14–23.
[10] F. Chiclana, E. Herrera-Viedma, S. Alonso, F. Herrera, A note on the estimation of missing pairwise preference values: A U-consistency based method, Int. J. Uncertain. Fuzziness Knowl. Based Syst. 16 (2008) 19–32.
[11] F. Chiclana, F. Herrera, E. Herrera-Viedma, Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations, Fuzzy Sets Syst. 97 (1998) 33–48.
[12] Y.C. Dong, X. Chen, F. Herrera, Minimizing adjusted simple terms in the consensus reaching process with hesitant linguistic assessments in group decision making, Inf. Sci. 297 (2015) 95–117.
[13] Y.C. Dong, E. Herrera-Viedma, Consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets and its use in the linguistic GDM with preference relation, IEEE Trans. Cybern. 45 (2015) 780–792.
[14] Y.C. Dong, W.C. Hong, Y.F. Xu, S. Yu, Selecting the individual numerical scale and prioritization method in the analytic hierarchy process: A 2-tuple fuzzy linguistic approach, IEEE Trans. Fuzzy Syst. 19 (2011) 13–25.
[15] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
[16] M. Fedrizzi, M. Brunelli, On the priority vector associated with a reciprocal relation and a pairwise comparison matrix, Soft Comput. 14 (2010) 639–645.
[17] E. Herrera-Viedma, S. Alonso, F. Chiclana, F. Herrera, A consensus model for group decision making with incomplete fuzzy preference relations, IEEE Trans. Fuzzy Syst. 15 (2007) 863–877.
[18] E. Herrera-Viedma, F. Chiclana, F. Herrera, S. Alonso, Group decision-making model with incomplete fuzzy preference relations based on additive consistency, IEEE Trans. Syst. Man Cybern. Part B Cybern. 37 (2007) 176–189.
[19] E. Herrera-Viedma, F. Herrera, F. Chiclana, M. Luque, Some issues on consistency of fuzzy preference relations, Eur. J. Oper. Res. 154 (2004) 98–109.
[20] J. Kacprzyk, Group decision making with a fuzzy linguistic majority, Fuzzy Sets Syst. 18 (1986) 105–118.
[21] H.S. Lee, T.W. K., Goal programming methods for constructing additive consistentcy fuzzy preference relations, in: B. Gabrys, R.J. Howlett, L.C. Jain (Eds.), Knowledge-Based Intelligent Information and Engineering Systems, 4252, Springer-Verlag, Berlin Heidelberg, 2006, pp. 910–916. KES 2006, Part II, LNAI.
[22] X.W. Liu, Y.W. Pan, Y.J. Xu, S. Yu, Least square completion and inconsistency repair methods for additively consistent fuzzy preference relations, Fuzzy Sets Syst. 198 (2012) 1–19.
[23] S. Miyamoto, Multisets and fuzzy multisets, Soft Computing and Human-Centered Machines, Springer, 2000, pp. 9–33.
[24] M. Mizumoto, K. Tanaka, Some properties of fuzzy sets of type 2, Inf. Control 31 (1976) 312–340.
[25] R.O. Parreiras, P.Y. Ekel, J.S.C. Martini, R.M. Palhares, A flexible consensus scheme for multicriteria group decision making under linguistic assessments, Inf. Sci. 180 (2010) 1075–1089.
[26] R.M. Rodríguez, B. Bedregal, B. Bustince, Y.C. Dong, B. Farhadinia, C. Kahraman, L. Martinez, V. Torra, Y.J. Xu, Z.S. Xu, F. Herrera, A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making, towards high quality progress, Inf. Fusion 29 (2016) 89–97.
[27] R.M. Rodríguez, L. Martínez, F. Herrera, A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets, Inf. Sci. 241 (2013) 28–42.
[28] R.M. Rodríguez, L. Martínez, F. Herrera, Hesitant fuzzy linguistic term sets for decision making, IEEE Trans. Fuzzy Syst. 20 (2012) 109–119.
[29] R.M. Rodríguez, L. Martínez, V. Torra, Z.S. Xu, F. Herrera, Hesitant fuzzy sets: State of the art and future directions, Int. J. Intell. Syst. 29 (2014) 495–524.
[30] T. Tanino, Fuzzy preference orderings in group decision making, Fuzzy Sets Syst. 12 (1984) 117–131.
[31] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (2010) 529–539.
[32] V. Torra, Y. Narukawa, On hesitant fuzzy sets and decision, in: Proceedings of the 2009 IEEE International Conference on Fuzzy Systems Fuzzy Systems, IEEE, 2009, pp. 1378–1382.
[33] M.M. Xia, Z.S. Xu, Managing hesitant information in GDM problems under fuzzy and multiplicative preference relations, Int. J. Uncertain. Fuzziness Knowl. Based Syst. 21 (2013) 865–897.
[34] M.M. Xia, Z.S. Xu, N. Chen, Some hesitant fuzzy aggregation operators with their application in group decision making, Group Decis. Negot. 22 (2013) 259–279.
[35] Y.J. Xu, Q.L. Da, L.H. Liu, Normalizing rank aggregation method for priority of a fuzzy preference relation and its effectiveness, Int. J. Approx. Reason. 50 (2009) 1287–1297.
[36] Y.J. Xu, Q.L. Da, H.M. Wang, A note on group decision-making procedure based on incomplete reciprocal relations, Soft Comput. 15 (2011) 1289–1300.
[37] Y.J. Xu, J.N.D. Gupta, H.M. Wang, The ordinal consistency of an incomplete reciprocal preference relation, Fuzzy Sets Syst. 246 (2014) 62–77.
[38] Y.J. Xu, F. Herrera, H.M. Wang, A distance-based framework to deal with ordinal and additive inconsistencies for fuzzy reciprocal preference relations, Inf. Sci. 328 (2016) 189–205.
[39] Y.J. Xu, K.W. Li, H.M. Wang, Consistency test and weight generation for additive interval fuzzy preference relations, Soft Comput. 18 (2014) 1499–1513.
[40] Y.J. Xu, K.W. Li, H.M. Wang, Incomplete interval fuzzy preference relations and their applications, Comput. Ind. Eng. 67 (2014) 93–103.
[41] Y.J. Xu, R. Patnayakuni, H.M. Wang, The ordinal consistency of a fuzzy preference relation, Inf. Sci. 224 (2013) 152–164.
[42] Z.S. Xu, Goal programming models for obtaining the priority vector of incomplete fuzzy preference relation, Int. J. Approx. Reason. 36 (2004) 261–270.
[43] D.J. Yu, W.Y. Zhang, Y.J. Xu, Group decision making under hesitant fuzzy environment with application to personnel evaluation, Knowl. Based Syst. 52 (2013) 1–10.
[44] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning – I, Inf. Sci. 8 (1975) 199–249.
[45] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.
[46] Z.M. Zhang, Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making, Inf. Sci. 234 (2013) 150–181.
[47] Z.M. Zhang, C. Wang, X.D. Tian, Multi-criteria group decision making with incomplete hesitant fuzzy preference relations, Appl. Soft Comput. 36 (2015) 1–23.
[48] B. Zhu, Z.S. Xu, Consistency measures for hesitant fuzzy linguistic preference relations, Fuzzy Syst. IEEE Trans. Fuzzy Syst. 22 (2013) 35–45.
[49] B. Zhu, Z.S. Xu, Regression methods for hesitant fuzzy preference relations, Technol. Econ. Dev. Econ. 19 (2013) S214–S227.
[50] B. Zhu, Z.S. Xu, J.P. Xu, Deriving a ranking from hesitant fuzzy preference relations under group decision making, IEEE Trans. Cybern. 44 (2014) 1328–1337.